Chapter X
Rotating Oscillators

X.1 A Brief Introduction

Here we are looking at oscillators located in a frame of reference rotating with an angular speed Ω. We shall consider the following cases: (i) A mass tethered with springs free to oscillate on an air-track, (ii) A simple pendulum, (iii) Two masses on an air-track tethered with springs. While these examples may appear 'contrived', rotating oscillators exist in many engineering situations, e.g., rotating machinery (of which there is a lot). Think of a flywheel in a car, if this is not perfectly balanced then it can be modelled as a mass attached to a balanced disk (via a stiff spring).

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X.2 Rotating Frames of Reference

The mathematics showing how to transform a force $\boldsymbol{F}$ from a stationary reference frame to a rotating frame has been presented in Chapter X where three *fictitious forces* were introduced to correctly describe the dynamics of a mass in the rotating frame. Here we summarize the results. These additional forces are illustrated in Fig.1 where we discuss the motion of a glider of mass m on an air track rotating about its centre.

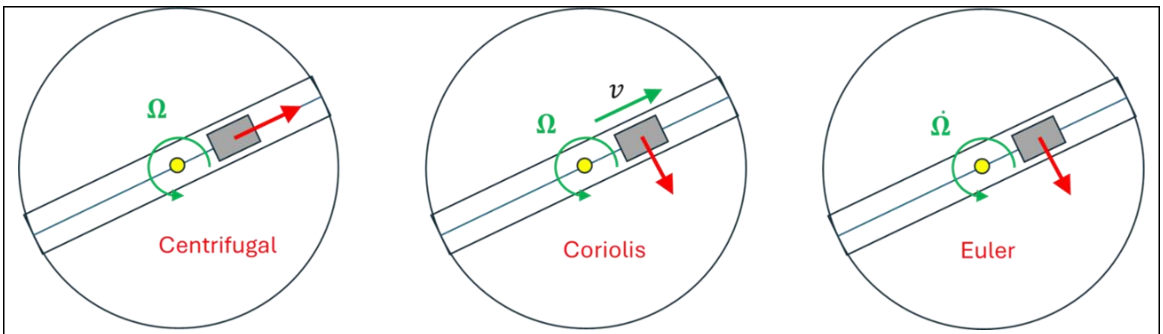

Figure 1. The three fictitious forces experienced in a rotating frame of reference. Note that the Euler force originates from a change in rotation speed.

When the glider is stationary, and the table is rotating with constant angular speed ω then the glider experiences a radially outwarddirected centrifugal force of magnitude $m\omega^2 r$ where r is its distance from the rotation centre. If the glider has a radial velocity v then it experiences the second *fictitious force*, the Coriolis force in the direction shown, with magnitude $2m\omega v$. Finally, if the table's angular speed is increasing, the glider experiences a third *fictitious force*, the Euler force of magnitude $m\dot{\omega}r$. Note that this is the force you may experience on a merry-go-round. Say this starts at rest and you are facing at right-angles to its radius and someone starts turning the merry-go-round, then you experience a force pushing you backwards, and you compensate by leaning forward.

Which of these forces are relevant for our discussion below? Well, only the centrifugal force, since we design the mechanics of the system to be constrained against both the Coriolis and Euler forces. For example, the glider on the air track is able to resist motion at right angles to its length (up to some limit of course).

X.3 Single Mass Single Spring

X.3.1 Symmetric Apparatus

The system is sketched in Fig.2(a) where we have a single glider of mass m and a couple of springs each with stiffness k. Clearly, we have a simple harmonic oscillator here since the springs provide a restoring force. The system ODE is simply, for any displacement x from the centre origin

$$m\ddot{x} = -2kx \qquad (1)$$

and the frequency of oscillation is

$$\omega^2 = \frac{2k}{m} \qquad (2)$$

When the supporting table is rotating with constant angular speed Ω then we have an additional centrifugal force of magnitude $m\omega^2 r$ Fig.2 (b)

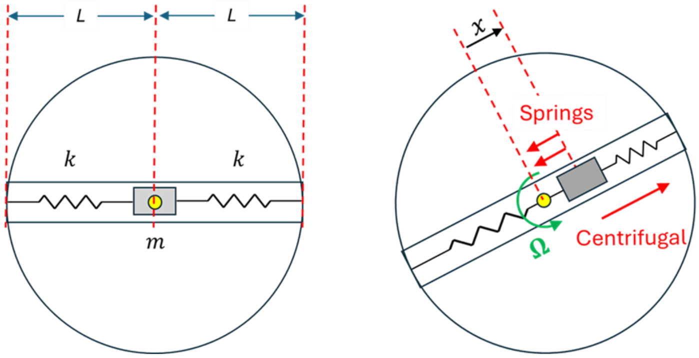

Figure 2. (a) Left shows a non-rotating glider attached to two springs. (b) Right shows displacement and forces when the air track is rotating.

The ODE now reads

$$m\ddot{x} = -2kx + m\Omega^2 x \quad (3)$$

$$= -(2k - m\Omega^2)x \quad (4)$$

and we see that the system still shows harmonic motion about the centre of rotation, but now with frequency

$$\omega^2 = \frac{2k}{m} - \Omega^2 \qquad (5)$$

so as the table rotation speed increases, the glider frequency decreases, until it becomes zero at a value $\Omega^2 = 2k/m$. This is a critical point, any increase in rotation speed would result in the centrifugal force exceeding any possible spring force and the apparatus would fail.

X.3.2 Offset Mass

If we have two unequal spring constants, then the equilibrium position of the glider is offset from the centre of rotation shown in Fig.3. The equation of motion of the glider is

$$m\ddot{x} = -(k_1 + k_2)x + L(k_2 - k_1) + m\Omega^2 x \qquad (6)$$

If there are no oscillations (perhaps they have been damped out) then the glider will be at rest at the equilibrium position

$$x_{Equ} = L\frac{k_2 - k_1}{k_1 + k_2} \qquad (7)$$

Let's look for a solution of the form

$$x(t) = A\cos(\omega t + \varphi) + C \qquad (8)$$

with initial conditions $x(0)$ and $\dot{x}(0) = 0$. Substitution of eq.8 into eq.7 and making use of the initial conditions, it is straightforward to obtain the solution

$$x(t) = [x(0) - C]\cos\omega t + C \qquad (9a)$$

with

$$C = \frac{L(k_2 - k_1)}{(k_1 + k_2 - m\Omega^2)} \qquad (9b)$$

$$\omega^2 = \frac{(k_1 + k_2)}{m} - \Omega^2 \qquad (9c)$$

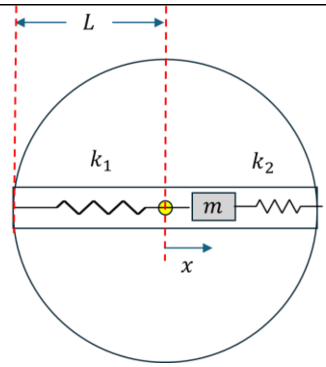

Figure 3. Equilibrium position of the mass is offset from the rotation centre due to two different spring constants.

This is all interesting and we can see some interesting things going on. First, it's clear that $(k_1 + k_2)$ is an important quantity. If this is much larger than $m\Omega^2$ then the equilibrium position approaches the value from eq.7, and the frequency of oscillation approaches the natural frequency of the mass-spring. Here the effect of the springs outweighs the effect of rotation.

Conversely for high values of Ω rotation becomes dominant, and as Ω approaches its critical value, both offset and oscillation period tend to infinity, see Fig.4.

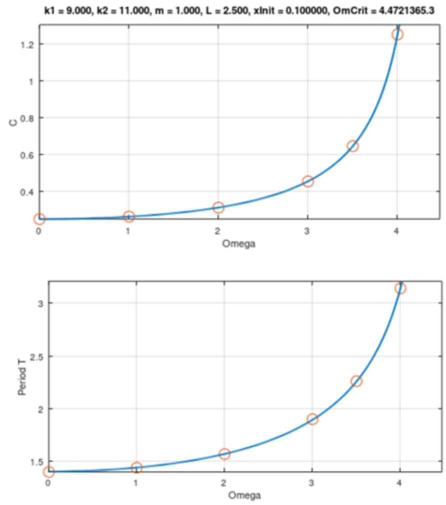

Figure 4.

Also the amplitude tends to minus infinity.

If we hold $(k_1 + k_2)$ constant but allow the difference to vary, then the offset increases with the difference $(k_2 - k_1)$, and of course increases as rotation speed is increased.

Finally, it's easy to check that the glider can never cross the centre of rotation but will oscillate between $x(0)$ and $2C - x(0)$ with an amplitude $x(0) - C$.

X.4 Pendulum on Rotating Table

This scenario is similar to the one just discussed where we have a simple pendulum suspended from a stand situated on a rotating table. Two configurations spring to mind, first where the suspension point is co-axial with the centre of rotation, and the second where it is offset. Both configurations are shown in Fig.5. Of course we need only to analyse the second configuration since the first is a special case. Here the long red dashed line shows the axis of rotation.

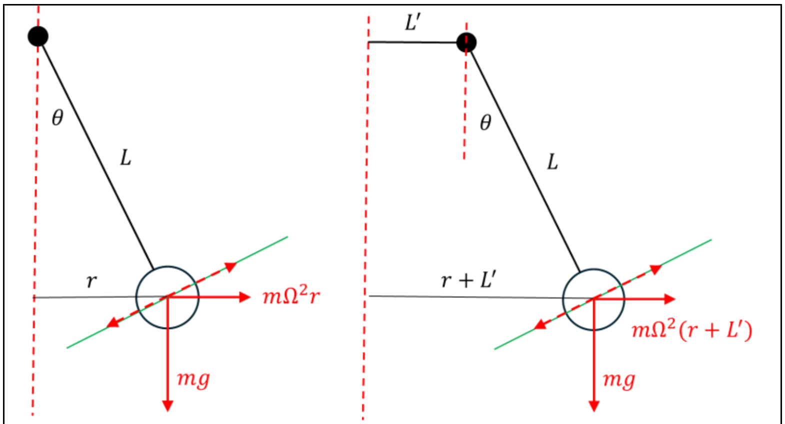

Figure 5. Two variants of a rotating pendulum: Left shows the pivot point lies on the axis of rotation. Right shows the case where the pivot point has some offset.

For the case on the right, we have the torques around the centre of rotation coming from the gravitational and centrifugal forces

$$\tau_{grav} = -mgL\sin\theta \qquad (10)$$

$$\tau_{cent} = (m\Omega^2 L\sin\theta + m\Omega^2 L')L\cos\theta \qquad (11)$$

Invoking the standard simplification $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ for small angles, we obtain the equation of motion of the offset pendulum.

$$\ddot{\theta} = \left(\Omega^2 - \frac{g}{L}\right)\theta + \left(\frac{L'}{L}\right)\Omega^2 \qquad (12)$$

from which we deduce the oscillation frequency

$$\omega^2 = \frac{g}{L} - \Omega^2 \qquad (13)$$

and we immediately see one effect of rotation is to reduce the pendulum's frequency. If we steadily increase the frequency then we shall reach the point where $\Omega = \sqrt{g/L}$ when the pendulum ceases to oscillate.

To obtain the analytical solution to eq.12 we proceed as in the case of the oscillating mass with the trial solution $\theta(t) = A\cos(\omega t + \varphi) + C$ which yields the following results

$$\theta(t) = [\theta(0) - C]\cos\omega t + C \quad (14a)$$

$$C = -\frac{\Omega^2 L'}{\Omega^2 L - g} \quad (14b)$$

which shows us that both the rotational speed and offset influence both the oscillation amplitude, and the average angular displacement of the bob. Eqs.14 shows that the pendulum behaviour is a rich mix of the rotational speed, the offset and the initial pendulum angle. Let us plan an investigation and take the offset L' as our independent variable. Two questions spring to mind, first how big does L' need to be so the pendulum just avoids passing back through the vertical. From eqs.14 we find the following relation

$$L' = \frac{1}{2}\theta(0)\left(\frac{g}{\Omega^2} - L\right) \quad (15)$$

Which fixes the required offset. Second, we ask is there an offset where the amplitude is zero? The answer is yes, and the value of offset L' is just the double of L' obtained from eq.15. Here's a plot of some solutions for various values of L'. The system parameters are $L = 2.5m$, $\Omega = 1.5\ rad/s$.

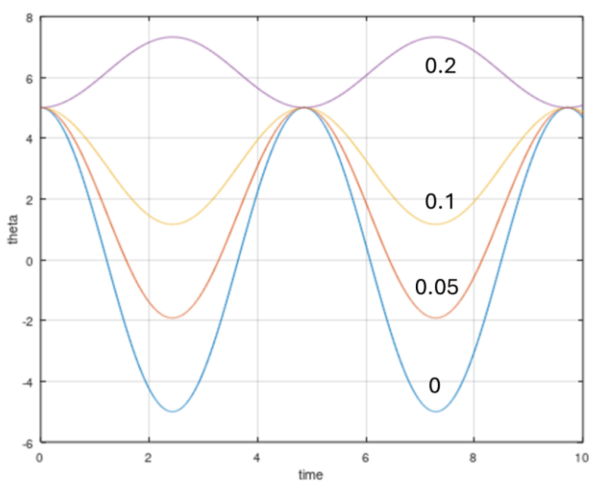

Figure 6. Trajectories of the rotating pendulum. Figures show the amount of offset (m).

You can see on how increasing the offset reduces the amplitude and lifts the mean pendulum angle, so it stops passing through the vertical. Ultimately the amplitude changes sign.

But there is a bit of a problem. In our rush to obtain a nice simple mathematically analytical solution, we have actually thrown away some important physics. Just think about a rotating pendulum which is not oscillating. Surely even without an offset, there will be cases where it will rotate with a constant offset angle. Let's see what has gone wrong.

We'll take the case of the pendulum without offset and write down its equation of motion *without assuming* $\sin\theta \approx \theta$, which is the origin of the problem.

$$\ddot{\theta} = \Omega^2 \sin\theta \cos\theta - \frac{g}{L}\sin\theta \qquad (16)$$

This can be solved for equilibrium solutions θ_{equ}

$$\theta_{equ} = \cos^{-1}\left(\frac{g}{\Omega^2 L}\right) \qquad (17)$$

and these can only exist if the pendulum rotates fast enough

$$\Omega > (g/L) \qquad (18)$$

So we have indeed found that the pendulum can have a constant offset without its axis of rotation having an offset. A diagram showing equilibrium angles as a function of rotation speed is shown in Fig.7.

Now we must ask ourselves if there could be any other consequences of the simplification on the physics. Well there might be since we have seen the limit on Ω in eq.18 before. Here we are told that the pendulum can have a constant offset if the rotation speed exceeds this limit, and our previous simplified analysis demonstrated that there could be no oscillations above this limit. Well, perhaps the simplified analysis has lost some more physics. So let's try to derive a non-simplified expression for the oscillation frequency.

We know that $\omega = \sqrt{-d\ddot{\theta}/d\theta}$ where this is evaluated at equilibrium angles θ_{equ} and from eq.16 we find

$$\omega^2 = \frac{g}{L}\cos\theta_{equ} - \Omega^2\left(2\cos^2\theta_{equ} - 1\right) \qquad (19)$$

then substituting eq.17 and after a little algebra we find

$$\omega^2 = \Omega^2 - \left(\frac{g}{L\Omega}\right)^2 \qquad (20)$$

This proves that the pendulum can oscillate when it is at an angle above the critical angle, but the oscillations have a different frequency characteristic than below the critical angle. Figure 8 provides a summary of this for a pendulum of length L = 2.5m.

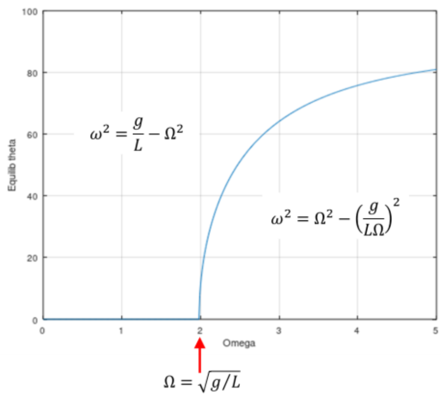

Figure 7. Equilibrium angles for rotation pendulum as a function of speed of rotation, showing a critical value.

We can now understand the behaviour of the oscillations starting from zero rotation speed. At first the pendulum oscillates about the vertical with a frequency $\sqrt{g/L}$ as expected. As the rotation speed increases, the pendulum's frequency decreases and becomes zero at the critical speed $\sqrt{g/L}$. At this point the pendulum starts to assume a constant offset with frequency close to zero, and as the rotation speed increases, so does the pendulum's frequency.

A plot of pendulum *period* following the above theory, with some experimental (simulation) data is shown in Fig.8

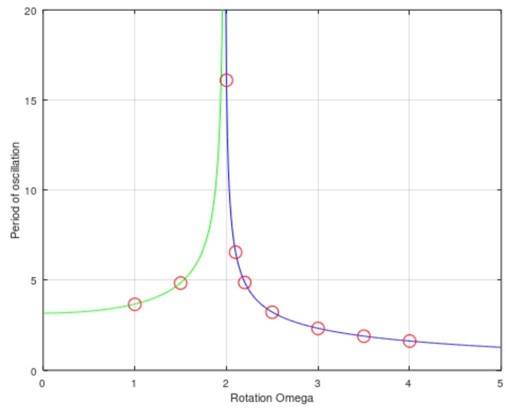

Figure 8. Dependence of oscillation period on rotation speed above and below the critical point. Circles show data from simulations.

The lines (theory) clearly show the different frequency behaviour below and above the critical rotation speed. A plot of oscillation amplitude dependence on rotation speed for some simulation investigations (Fig.9) shows a dramatic increase in amplitude as Ω passes its critical value.

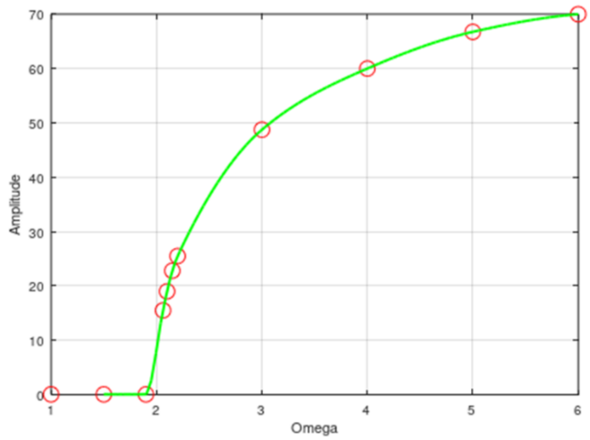

Figure 9. Dependence of pendulum oscillation angle on rotation speed. Circles show simulation data points, the green line is an interpolation.

It may also be useful to look at some actual simulated trajectories for a number of Ω values. Striking is the non-harmonic form of the trajectories. The period is large near the critical point and decreases as Ω is increased. Interestingly the pendulum returns to its vertical position for all trajectories, and for higher values of Ω its average angle is almost horizontal.

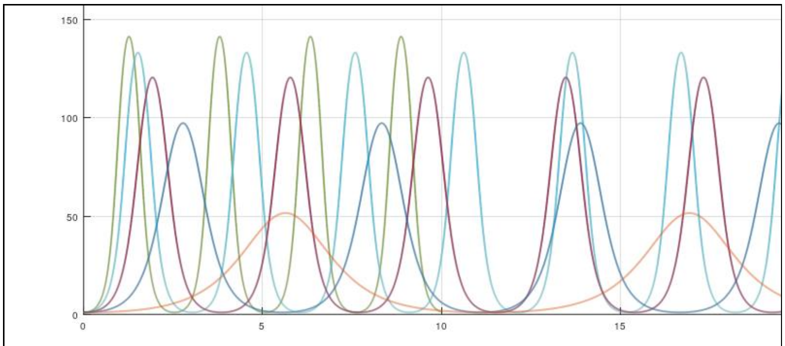

Figure 10. Simulation trajectories for various rotation speeds. Need to label figure.

X.5 Rotating Double Mass with Three Springs

Let's build a more complex system with two equal masses and three equal springs arranged like this. Again there is rotation about the centre with a specified angular speed Ω, the axis of rotation is shown by the dashed line.

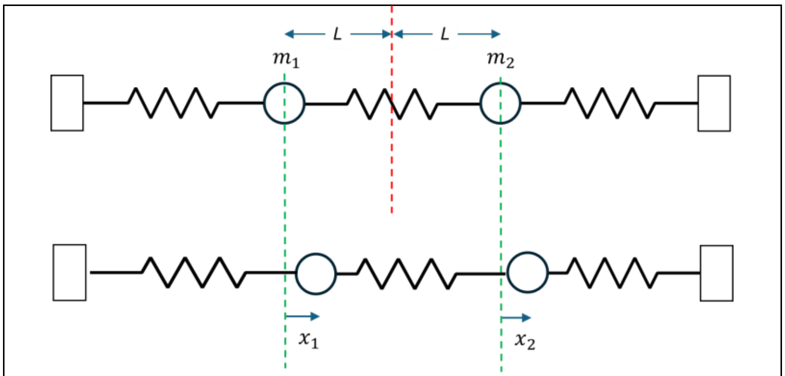

Figure 11. Configuration of two masses and three equal springs. Red dashed line shows axis of rotation, green dashed lines show initial location of masses when there is no rotation.

Writing the displacements of the masses about their static equilibrium positions $\pm L$ as x_1 and x_2 it's straightforward to obtain the equations of motion

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) - m\Omega^2(L - x_1)$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) + m\Omega^2(L + x_2) \qquad (21)$$

Let's attempt to analyse this system by proceeding in a way we understand, looking for *normal modes* of vibration of a similar system without rotation. We want to know if normal modes can exist with the presence of rotation. It's useful first to conduct a short thought experiment to crystallize (or challenge) our thinking. Let's assume the existence of normal modes shown in Fig.12, *the symmetric* mode where the masses move in opposite directions, and the *antisymmetric* mode where they move in the same direction. Now for the symmetric mode, both masses will have the same distance from the axis of rotation, so the centrifugal terms will be equal and opposite. So the assumption that this mode exists is a good one. Now for the antisymmetric mode, since the masses are at different distances from the rotation axes, then their centrifugal forces will be different. The question is, can the springs compensate for this. We suspect they can but need to perform some analysis to prove this.

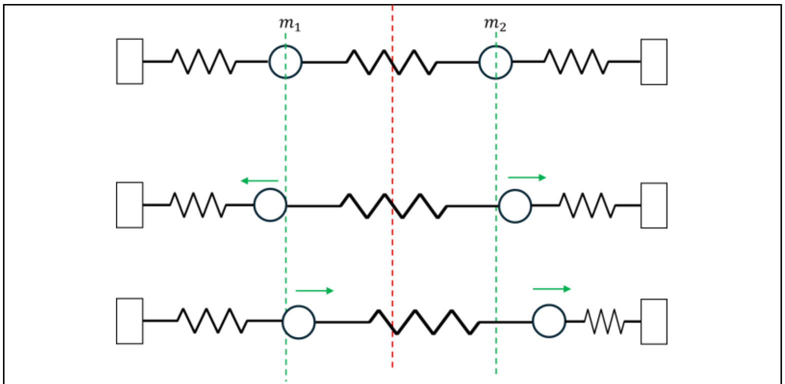

Figure 12. Normal modes. Top shows equilibrium positions, centre shows the symmetric mode, bottom shows the antisymmetric mode.

Let's start by writing down the equation for the possible *symmetric* mode after a little cleaning up

$$m(\ddot{x_1 - x_2}) = -[3k - m\Omega^2](x_1 - x_2) - 2m\Omega^2 L \qquad (22)$$

and for the possible *antisymmetric* mode,

$$m(\ddot{x_1 + x_2}) = -[k - m\Omega^2](x_1 + x_2) \qquad (23)$$

These equations do resemble our usual normal mode equations (see Chapter X) and we recognize respective high and low oscillation frequencies

$$\omega_{sym}^2 = \frac{3k}{m} - \Omega^2 \quad (24a)$$

$$\omega_{anti}^2 = \frac{k}{m} - \Omega^2 \quad (24b)$$

which is encouraging. But the normal mode equations themselves are not symmetric, we have a constant forcing term $-2m\Omega^2$ in eq.22. But these two equations are decoupled, so they can be solved independently. To do this, let's introduce two new variables $z_{Sym} = x_1 - x_2$ and $z_{ASym} = x_1 + x_2$ whose names give away what we have in mind. Let's re-write eqs.22,23 as

$$m\ddot{z}_{Sym} = -[3k - m\Omega^2]z_{Sym} - 2m\Omega^2 L \quad (25)$$

$$m\ddot{z}_{ASym} = -[k - m\Omega^2]z_{ASym} \quad (26)$$

Let's take eq.25 and try a solution of the form

$$z_{Sym}(t) = A\cos(\omega t + \varphi) + C \quad (27)$$

Substituting into eq.25, after a little algebra we find the following results

$$C = -\frac{2m\Omega^2 L}{(3k - m\Omega^2)} \quad (28)$$

and

$$\omega_{Sym}^2 = \frac{3k}{m} - \Omega^2 \quad (29)$$

we know the latter well. Now let us invoke the initial conditions $z_{Sym}(t=0) = z_{Sym}(0)$ and $\dot{z}_{Sym}(t=0) = 0$ which helps us find the phase φ and amplitude A. The final solution becomes

$$z_{Sym}(t) = \left(z_{Sym}(0) - C\right)\cos\left(\omega_{Sym} t + \varphi\right) + C \quad (30)$$

Proceeding in a similar fashion for $z_{ASym}(t)$ we find the solution

$$z_{ASym}(t) = z_{ASym}(0)\cos\omega_{ASym} t \quad (31)$$

With

$$\omega_{ASym}^2 = \frac{k}{m} - \Omega^2 \qquad (32)$$

So we have found the existence of two normal modes with high and low frequencies as expected. How do we select the initial displacements to excite either the symmetric or antisymmetric modes? Eq.30 gives us the condition for the non-existence of the symmetric mode

$$z_{Sym}(t) = 0 \Rightarrow z_{Sym}(0) = C$$

$$x_2(0) = x_1(0) - C \qquad (33)$$

and similarly for the symmetric mode we have for the non-existence of the antisymmetric mode

$$z_{ASym}(t) = 0 \Rightarrow z_{ASym}(0) = 0$$

$$x_2(0) = -x_1(0) \qquad (34)$$

Here the idea is that we have free choice of $x_1(0)$ which then determines the value of $x_2(0)$ for both cases.

Finally, what are the equilibrium states (e.g., when any oscillations have been damped). From eqs.30,31. These are simply

$$\bar{x}_1 = C/2$$

$$\bar{x}_2 = -C/2 \qquad (35)$$

The equilibrium configuration is shown in Fig.13

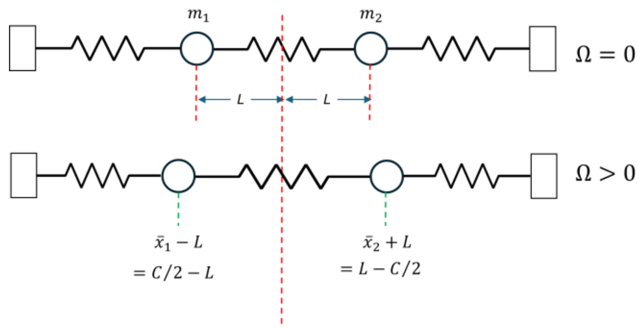

Figure 13. Equilibrium states: (a) Top without rotation, (b) Bottom with rotation.

From eqs.30,31 we can extract expressions for $x_1(t)$ and $x_2(t)$

$$x_1(t) = \left[x_1(0) - \frac{C}{2}\right]\cos\omega t + \frac{C}{2} \qquad (36)$$

$$x_2(t) = \left[x_2(0) + \frac{C}{2}\right]\cos\omega t - \frac{C}{2} \qquad (37)$$

where there seems to be some nice symmetry going on. The initial conditions determine, of course whether we excite either pure mode, or a superposition of two modes in various quantities. Of course this mix will always contain components of two frequences ω_{Sym} and ω_{ASym}.

Let's have a look at an example of symmetric and antisymmetric modes for a system with parameters m_1=1kg, k = 39.478 N/m, L = 0.5m, Ω=5rad/sec. The symmetric solution is shown in Fig.14. Note that the trajectories are shown about the rotating equilibrium points. We applied initial conditions $x_1(0) = -0.4, x_2(0) = 0.4$. The amplitudes of oscillation are $A_1 = x_1(0) - C/2$ and $A_2 = x_2(0) + C/2$ which are ± 0.2662 respectively.

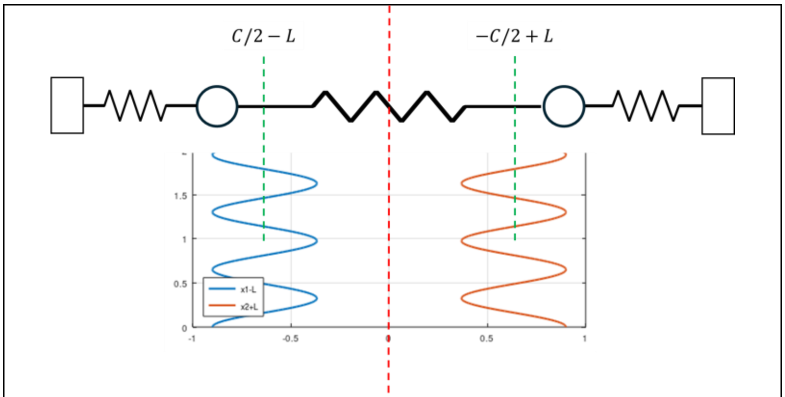

Figure 14. Example of system behaviour for the symmetric mode.

The antisymmetric mode is shown in Fig.15. We applied initial conditions $x_1(0) = 0.1$, and $x_2(0) = x_1(0) - C = 0.3676$. The amplitudes of oscillation are $A_1 = x_1(0) - C/2$ and $A_2 = x_2(0) + C/2$ which are ± 0.2338 respectively.

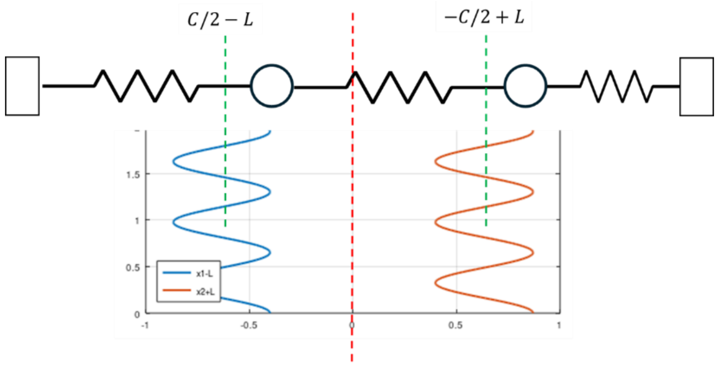

Figure 15. Example of system behaviour for the antisymmetric mode.

Results of investigations of the effect of Ω on the equilibrium locations of the gliders for symmetric and antisymmetric modes are shown in Fig.16.

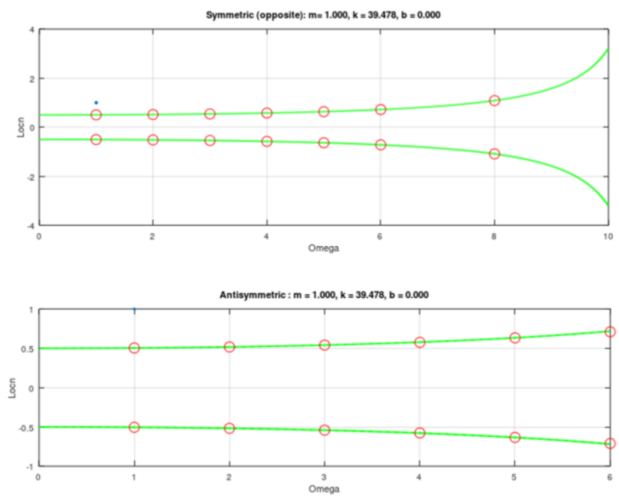

Figure 16. Simulation results of equilibrium locations for symmetric and antisymmetric solutions (circles) with theoretical solution.

Also effects of Ω on the glider period for both modes are shown in Fig.17. This material would make a good investigation.

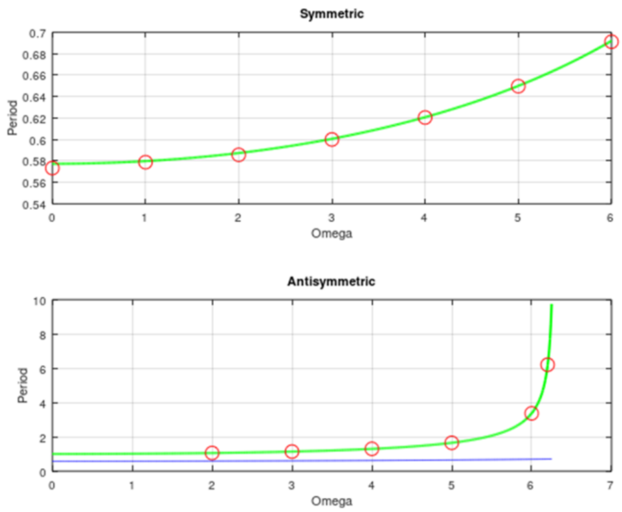

Figure 17. Simulation results of oscillation periods for symmetric and antisymmetric solutions (circles) with theoretical solution.