Chapter X
Kapitza Pendulum

X.1 A Brief Introduction

Galileo's pendulum has been around for hundreds of years, and we are all quite familiar with its behaviour, oscillating around an upper suspension point. What is truly amazing is that when the suspension point is made to oscillate *vertically* with a frequency much higher than the pendulum's natural frequency, then the pendulum can oscillate but in a vertical orientation, above its pivot point. This is known as Kapitza's pendulum, and the PhysLab incarnation is shown in Fig.1.

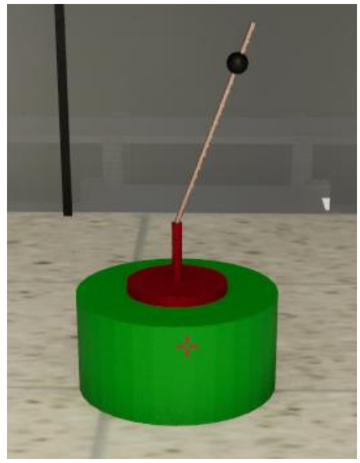

Figure 1

Do not confuse this with a vertical pendulum mounted on a cart which can move horizontally, such as a Segway, the stabilization is produced by a different mechanism. The material below is quite mathematical, though we shall attempt to emphasise the physics.

X.2 Equation of Motion

X.2.1 Simple Starting point considering torques.

Here we take a simple approach to deriving the equation of motion, a more rigorous approach is presented at the end of the chapter using Lagrange's formulation. For the moment consider the two pendulums shown in Fig.2 .

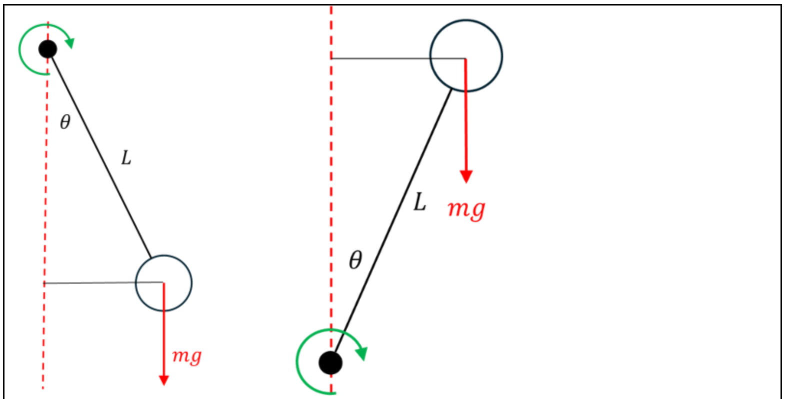

Figure 2.

Let's take Galileo's pendulum on the left. The downward force on the bob due to gravity produces a torque around the pivot point

$$mgL\sin\theta \qquad (1)$$

and so the equation of motion for the bob's rotation becomes

$$mL^2\ddot{\theta} = -mgL\sin\theta \qquad (2)$$

which simplifies to the expression you know well

$$\ddot{\theta} = -\frac{g}{L}\sin\theta \qquad (3)$$

which of course shows that gravity acts to reduce θ at this point. For Kapitza's pendulum on the right we have a similar result

$$\ddot{\theta} = \frac{g}{L}\sin\theta \qquad (4)$$

showing that gravity increases θ at this point, the pendulum seeks to hang down.

How let's introduce a vertical oscillation of the pivot point, Fig.3 where $a(t)$ is a *small* and *high frequency* oscillating displacement.

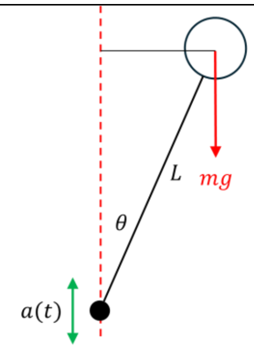

Figure 3.

Now we can jump to a non-inertial reference frame associated with the oscillating axis, we can see that gravity g is supplemented by an additional acceleration due to the oscillating (and therefore accelerating pivot). We can then replace g in the equation of motion with

$$-g - \ddot{a}(t) \qquad (5)$$

This expression is general, if we take harmonic excitation of the form $a(t) = A\cos\Omega t$ then the augmented gravity becomes

$$-g + \Omega^2 A\cos\Omega t \qquad (6)$$

and inserting into eq.4 our equation of motion becomes

$$\ddot{\theta} = \frac{g}{L} \sin\theta - \Omega^2 \frac{A}{L} \cos\Omega t \sin\theta \qquad (7)$$

This is the equation we must solve.

X.2.2 A Tale of Two Time scales

There are tools which we may draw from out mathematical armoury to deal with eq.7 but things get a little easier and more interesting if we look at the physics of the Kapitza pendulum. Long before Kapitza did his analysis, there were lots of experimental results available, including graphs of the bob's motion. We shall see the importance of experimental work in understanding physics phenomena.

The basic experimental result is shown in Fig.4 which comes from a PhysLab simulation, and which may be directly observed in the simulation

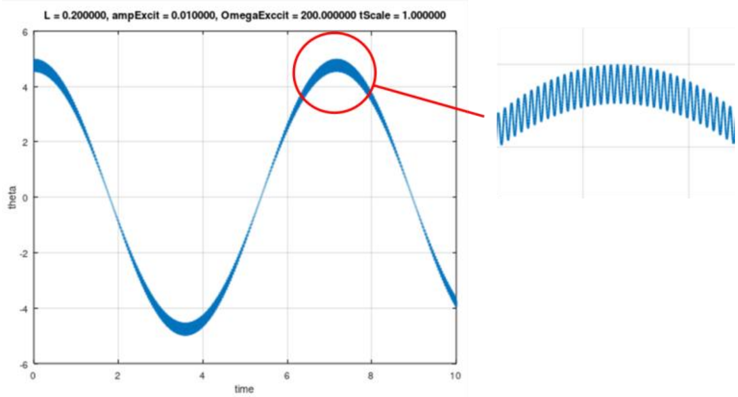

Figure 4.

The bob has been given an initial displacement from the vertical of 5^o and is seen to be merrily oscillating around the vertical with this amplitude, and a certain frequency of vibration (close to its natural frequency). But there is an additional oscillation superposed on this, and this has two important properties (i) it has a much higher frequency and (ii) it has a much smaller amplitude. Both of these properties are essential to understanding how *small-amplitude high-frequency vertical oscillations of the pivot can stabilize the vertical pendulum*. The second of these properties translates to the condition $A \ll L$ since L sets the length scale of the problem.

Let's start work on eq.7 with what we have just said in mind. Let's try a solution of the form

$$\theta(t) = \theta_0(t) + \varphi(t) \qquad (8)$$

where $\theta_0(t)$ is *slow* and *large* and $\varphi(t)$ is *fast* and *small*. First, we have

$$\sin(\theta_0 + \varphi) \approx \sin\theta_0 + \varphi\cos\theta_0 \qquad (9)$$

by Taylor expansion or using trig. identities. Substituting in eq.7 we have

$$\ddot{\theta}_0 + \ddot{\varphi} + \frac{\Omega^2 A}{L}\cos\Omega t\sin\theta_0 + \frac{\Omega^2 A}{L}\cos\Omega t\,\varphi\cos\theta_0 - \frac{g}{L}\sin\theta_0 - \frac{g}{L}\varphi\cos\theta_0 = 0 \quad (10)$$

Now at high frequencies we have $\Omega^2 A \gg g$ so we can ignore the last two terms involving g. We can also ignore $\ddot{\theta}_0$ since θ_0 is slow and so its second derivative is small, compared with $\ddot{\varphi}$ which is large since φ is fast. We can also ignore the 4th term which contains φ which is small. We therefore end up with

$$\ddot{\varphi} \approx -\frac{\Omega^2 A}{L}\sin\theta_0\cos\Omega t \qquad (11)$$

Now since θ_0 is slow, we can take $\sin\theta_0$ as pretty much constant over one cycle of φ so we can integrate eq.11 directly to obtain an expression for the fast variable,

$$\varphi(t) = \frac{A}{L}\cos\Omega t\sin\theta_0 \qquad (12)$$

Before moving on, let's do a sanity check, recalling our assumptions and checking that eq.12 agrees with these. We assumed that $A \ll L$ and that φ was small. This is clearly born out in the first factor of eq.12.

Now let's return to eq.10 and now extract the solution for the slow variable θ_0. We proceed by taking a *time average* of the quantities over a period of time equal to the period of the fast oscillations. The idea is that we don't expect the fast oscillations to affect the slow oscillations in any significant detail since these are small. First we remove the term $\ddot{\varphi}$ since this is sinusoidal with an average of zero. The third term is removed since $\cos\Omega t$ has a zero average. The final term is removed since this contains φ which has an average of zero. The fourth term contains $\cos\Omega t$ and φ which varies as $\cos\Omega t$. We end up with

$$\ddot{\theta}_0 + 0 + 0 + \frac{\Omega^2 A}{L}\cos\theta_0 \frac{A}{L}\sin\theta_0\langle\cos^2\Omega t\rangle - \frac{g}{L}\sin\theta_0 - 0 = 0 \quad (13)$$

and finally

$$\ddot{\theta}_0 + \frac{\Omega^2 A^2}{2L^2}\cos\theta_0 \sin\theta_0 - \frac{g}{L}\sin\theta_0 = 0 \quad (14)$$

This equation is formidable since it cannot be integrated directly, but we can introduce the usual small-angle approximations

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3 \qquad \sin\theta\cos\theta \approx \theta - \frac{2}{3}\theta^3 \quad (15)$$

where we have kept terms up to order 3. We finally end up with

$$\ddot{\theta}_0 + \left(\frac{\Omega^2 A^2}{2L^2} - \frac{g}{L}\right)\theta_0 + \left(\frac{1}{6}\frac{g}{L} - \frac{\Omega^2 A^2}{3L^2}\right)\theta_0^3 = 0 \quad (16)$$

X.2.3 Oscillation Frequencies and Bifurcation Curves

Our eq.16 provides some very useful information about our system. If we assume a solution of the form, $\theta(t) = A_{Slow}\cos\omega t$ then we find the slow frequency of oscillation is

$$\omega^2 = \frac{\Omega^2 A^2}{2L^2} - \frac{g}{L} \quad (17)$$

and there is a limit on ΩA for harmonic oscillations

$$(\Omega A)^2 > 2gL \quad (18)$$

To check this against our original assumptions we can better write this as

$$\frac{\Omega^2 A}{g}.\frac{A}{L} > 2 \quad (19)$$

Looking at the first term, we assumed that $\Omega^2 A > g$ so this term is large, and for the second term we assumed $A \ll L$ so this term is small. This means that to obtain stable oscillations, we expect to need large values of Ω which satisfy

$$\Omega > \frac{\sqrt{2gL}}{A} \qquad (20)$$

We can also extract the bifurcation equation from eq.16 looking for equilibrium solutions at a general angle. We find a pitchfork bifurcation, but the branches are unstable, so there are only two stable points for the Kapitza pendulum, for $\theta = 0, \pi$.

$$\theta_{0Equ}^2 = -\frac{(3\Omega^2 A^2 - 6gL)}{(2\Omega^2 A^2 - gL)} \qquad (21)$$

A similar analysis can be performed about the lower stable point, the Galilean pendulum. The frequency of low-amplitude oscillations here is

$$\omega^2 = \frac{\Omega^2 A^2}{2L^2} + \frac{g}{L} \qquad (22)$$

Measured frequencies as a function of excitation amplitude A for both upper and lower solutions from simulations in PhysLab are shown in Fig.5 together with the theoretical expressions from equ.17 and equ.22.

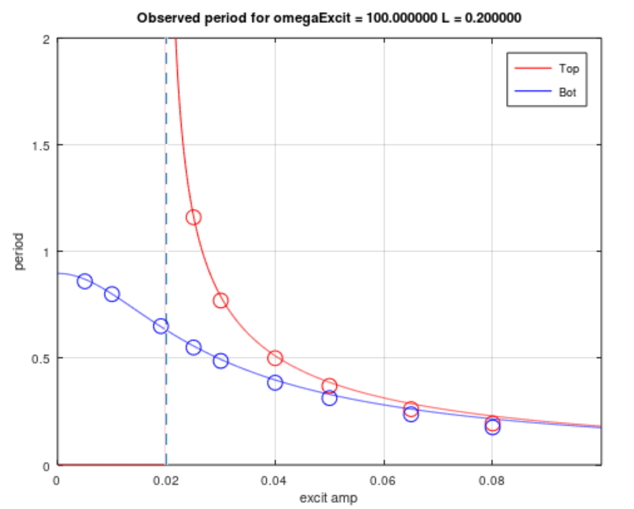

Figure 5.

We also show the results of a simulation study for parameters \l = *0.2, A = 0.02*, $\Omega = 110$ for an initial displacement of 1 degree. We model the low frequency component as a cosine curve of this amplitude and period determined by eq.17 and subtract this from the solution to reveal the fast component (green). It's amplitude agrees with eq.12 to within an order of magnitude.

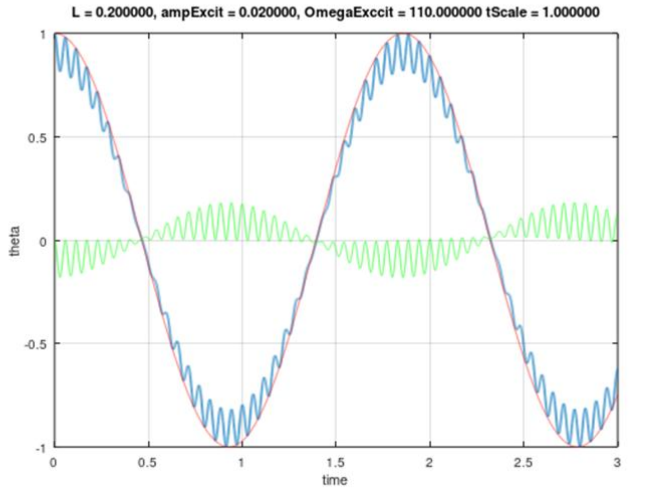

Figure 6.

Our analysis so far has been concerned with small-amplitude oscillations. Now we turn to exploring to what extent we can understand large-amplitude oscillations.

X.3 Large Amplitude Solutions

Our analysis at the end of section X.2.2 and in section X.2.3 focussed on small-amplitude oscillations. Here we are interested in extending the results obtained. The question is simple, we know the frequency of the small-amplitude oscillations, but we were not given any information about the *amplitude* of those oscillations. Not surprising since for a linear oscillator, the amplitude depends on the initial conditions. We can do better here, since the Kapitza pendulum is certainly nonlinear.

We turn to formulating our system in terms of *energy*. Why energy? Well unlike local approximations, energy equations do not have to involve approximations and are able to give us a more *global* view of our system behaviour.

Let's reproduce eq.14 to make our argument clear.

$$\ddot{\theta}_0 + \frac{\Omega^2 A^2}{2L^2}\cos\theta_0 \sin\theta_0 - \frac{g}{L}\sin\theta_0 = 0 \quad (23)$$

As we mentioned we cannot integrate this directly since we have transcendental functions present. But in situations like this where we have to integrate a term like $\ddot{\theta}_0$ we can use the powerful equivalence

$$\ddot{\theta}_0 = \frac{d}{d\theta_0}\left(\frac{1}{2}\dot{\theta}_0^2\right) \qquad (24)$$

So eq.23 becomes

$$\frac{1}{2}\dot{\theta}_0^2 + \int\left(\frac{\Omega^2 A^2}{2L^2}\cos\theta_0 \sin\theta_0 - \frac{g}{L}\sin\theta_0\right)d\theta_0 = 0 \qquad (25)$$

In effect we have transformed integration over time into integration over θ_0. This is straightforward where we obtain

$$\frac{1}{2}\dot{\theta}_0^2 + \frac{\Omega^2 A^2}{2L^2}\frac{\sin^2\theta_0}{2} + \frac{g}{L}\cos\theta_0 = cst1 \quad (26)$$

The first term reminds us of kinetic energy so let's make this explicit, remembering that linear velocity is just $L\dot{\theta}_0$

$$\frac{1}{2}mL^2\dot{\theta}_0^2 + \frac{m\Omega^2 A^2}{4}\sin^2\theta_0 + mgL\cos\theta_0 = cst2 \quad (27)$$

So this is an expression for the total energy of our system, the first term is the kinetic energy, the third term is the usual gravitational potential energy of a free pendulum (*A=0*), but the second term is an extra potential energy term due to the excitation.

So we have discovered an effective *potential* (not potential energy, we remove *m*) which describes the large-amplitude motion of the pendulum, as follows,

$$U(\theta_0) = \frac{\Omega^2 A^2}{4}\sin^2\theta_0 + gL\cos\theta_0 \qquad (28)$$

which will define the *limits* of the oscillation angle. To make this a little more concrete, let's take the case of a pendulum of length L=0.8 driven by $\Omega = 230$ and A = 0.04 then the plot of potential against angle is shown in Fig.7 We see the potential well keeps the bob well constrained to a limit of around 80 degrees. This is a *huge* angle

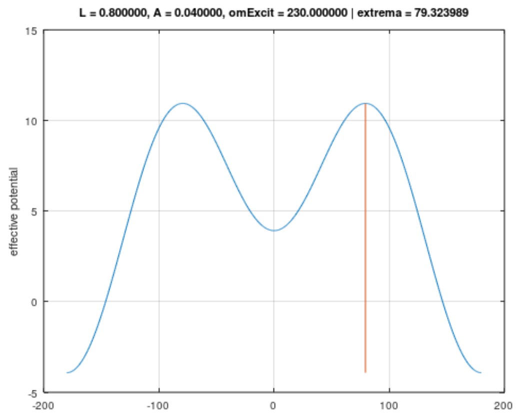

Figure 7.

There are three parameters which define this potential, L, A, and Ω. In investigating, or designing, a system, we would normally fix L which defines the length scale for the system, then see how the potential varies with driving amplitude and driving frequency. Figure 8 shows a plot of potential for L=*0.8m* and $\Omega = 230\ \mathrm{rad/s}$. As A is increased the hills surrounding the valley become larger and further apart. There is also a critical drive amplitude which must be met for the vertical oscillation to be stable.

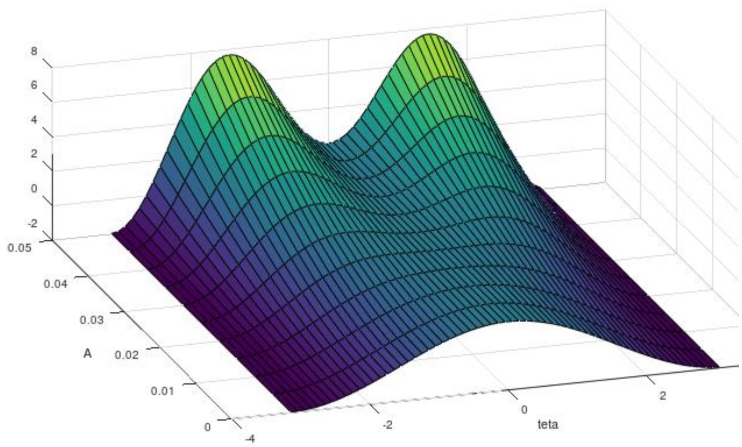

Figure 8.

Looking mathematically for the peaks of the potential hill, the first derivative of $U(\theta_0)$ tells us that the peaks are located at

$$\theta_0 = \cos^{-1}\left(\frac{2gL}{\Omega^2 A^2}\right) \qquad (29)$$

which leads to the following condition

$$(\Omega A)^2 > 2gL \qquad (30)$$

which we have seen before (eq.18).

X.4 Running Orbits