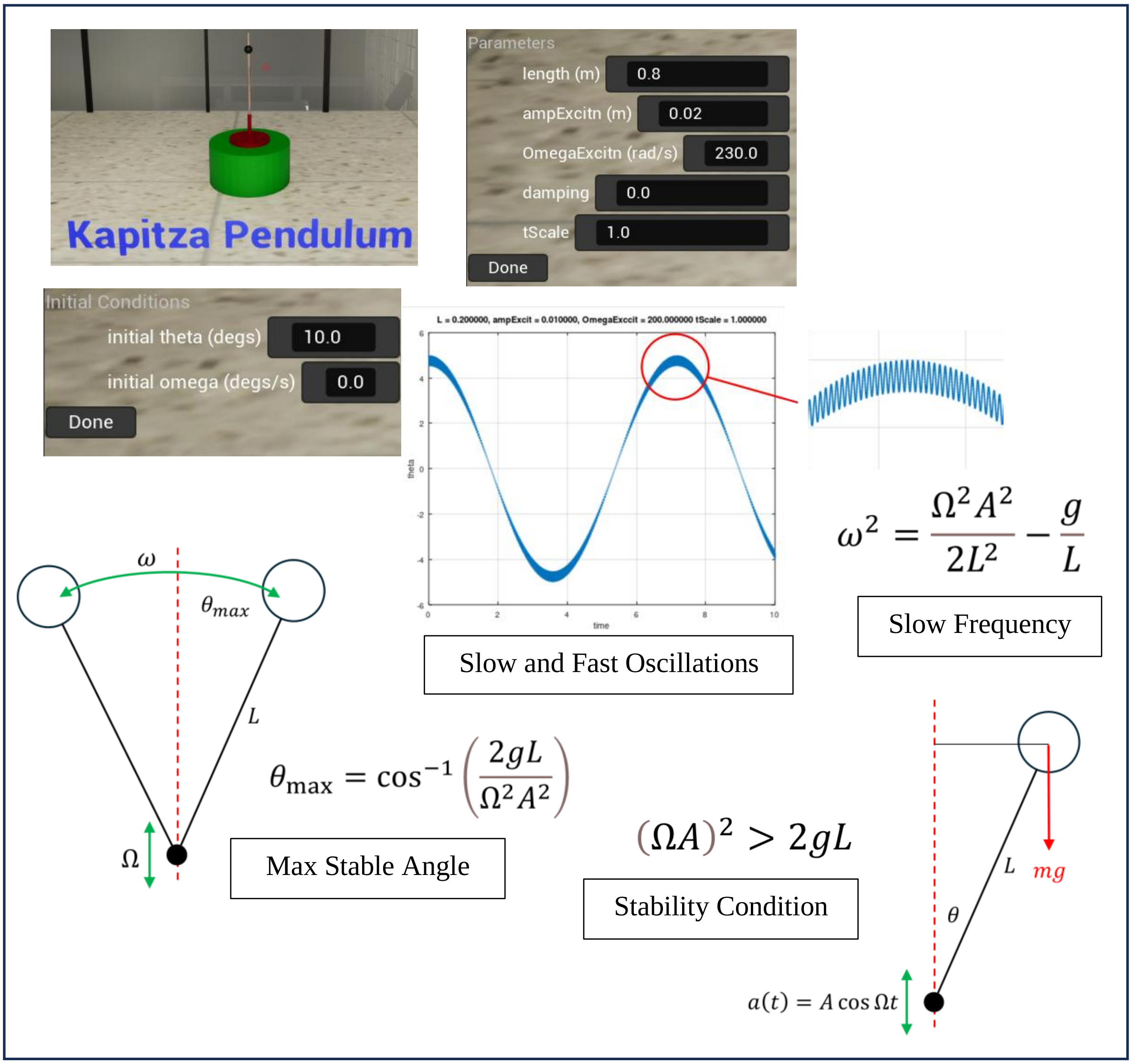
Chapter X
Investigations

X.1 Kapitza's Pendulum



X.1.1 First Experiments

(1) Preparation

In this experiment the base of the pendulum oscillates vertically at a high frequency, and the pendulum becomes inverted and performs angular oscillations around the vertical.

(a) What would you expect to see if the vertical oscillation frequency is steadily reduced?

(b) Use the given values for parameters L ‘length’ and A ‘ampExcitn’ to calculate the critical angular rotation speed Ω.

(c) Calculate the maximum angle allowed for stable oscillations for rotational speeds 230, 210 and 200 r/s.

(2) Guided Data Collection

(a) Activate the experiment and press **P** to bring up the parameters. Check that Ω ‘OmegaExcitn’ is set to 230. Press **Done**.

(b) Press **I** to bring up the initial conditions. Make sure ‘initial theta’ is set to 10 degrees (not a small value). Press **Done**.

(c) Press **F1** to run the experiment. Convince yourself that the bob displays small-amplitude high-frequency vertical oscillations superposed on large-amplitude low-frequency sideways oscillations.

(d) Observe the slow oscillations for a few cycles and note down an estimate of their period. Compare this with the period readout on the HUD.

(e) Press **F3** to stop the experiment and restore to the initial conditions you have set.

(f) Now press **P** to bring up the parameters and set Ω to 210.

(g) Now press **F1** to run the experiment for a few cycles and again note down a period estimate also the value from the HUD.

(h) Repeat appropriate steps to gather data for Ω set to 200 (which is just above the critical Ω). When you are done press **X** to disengage the experiment.

(3) Looking at the logged data in Octave.

(a) Press **O** to open up Octave and you will see plots of θ and $\dot{\theta}$ against time. Make sure you can see both slow and fast oscillations superposed.

(b) Take measurements on the various sections of the θ plot to calculate the oscillation period for each value of Ω. Compare these with your values recorded earlier. Close down Octave

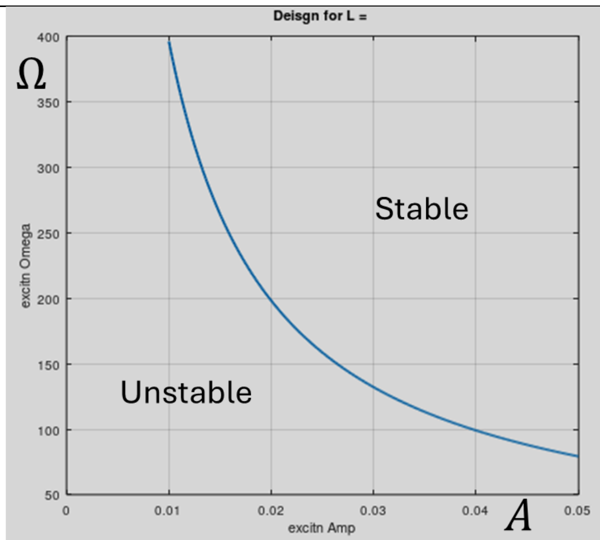
(c) Navigate to the folder **Octave | Kapitza** and open the Octave script '**PeriodChecks.m'**. Enter your data Ω, T in the area indicated.
(d) When you run the script, your experimental data will be plotted on the theoretical curve obtained from the 'Slow Frequency' expression in the box above.
(e) Think carefully about what you see. Can you explain why some experimental points look OK, but one certainly is not?

(4) Repeating for a small initial angle
You probably realized that the lack of agreement was because the initial angle chosen was large, and the small-amplitude theory does not handle this. So repeat (3) with an initial angle of 1 degree and see if you get any better agreement. You must set this angle in the initial conditions dialogue box.

(5) Pause for reflexion
Make short notes for yourself on what you have learned so far.

X.1.2 Investigations

(6) Independent mini-Investigation - 1
The plot below shows the stability region in $A - \Omega$ space. Choose pairs of A and Ω where the product $A\Omega$ is about the same. Are there any differences in the form of the pendulum trajectory?



(7) Independent mini-Investigation - 2

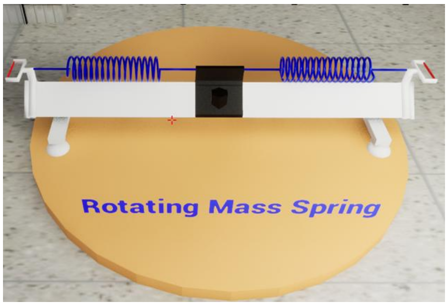
Gather some data for values of Ω below the critical value. Think carefully and choose some interesting values for Ω. Use Octave plots to interpret what is going on.

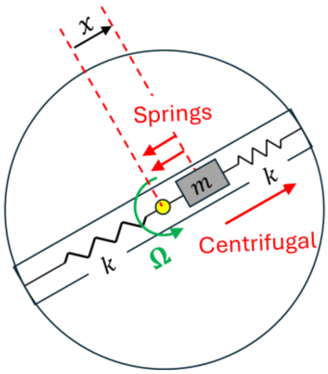
(8) Investigation

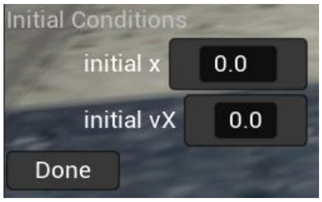
There is an extensive literature on the Kapitza pendulum including numerical simulations which PhysLab could replicate. Folk have found interesting solutions such as (i) ‘looping’ behaviour where the bob executes repeated rotation, (ii) superposition of rotations and oscillations, and of course, (iii) *chaos*.

In this sort of situation where there are several parameters, the trick is to choose one which is likely to lead to these interesting solutions. You will probably need to try a few things out before committing to using one principal parameter. But it’s probably best *not* to think of initial conditions as a parameter, these should be held fixed, at least during the planned investigation.

X.2 Rotating Spring-Mass Oscillator







$$\omega^2 = \frac{2k}{m} - \Omega^2$$

oscillation frequency

$$m\ddot{x} = -2kx + m\Omega^2 x$$

equation of motion

X.2.1 First Experiments

(1) Observing the Oscillations without Rotation

(a) When the table is not rotating, the mass will oscillate. How do you expect the behaviour of the mass to change as the table is made to rotate, and its angular velocity progressively increased?

(b) Press **P** and check that the parameters are set to their default values as shown in the box.

(c) Press **I** and set initial X to 0.25

(d) Press **F1** to run and observe the oscillations and make a mental note of their period from the HUD.

(e) Press **F2** to reset then press **P** and increase the values of k1 and k2, but keep them equal
(f) Press **F1** and observe the oscillations. Has the period changed as you would expect?
(g) Press **F2** then press **P** and restore k1 = k2 = 10. Then press **X** to disengage the experiment.

(2) Observing the effect of Rotations

Here you are invited to make *direct observations* of the oscillator as we steadily increase the rotation speed. No need to record any data here.

You can change your viewpoint by pressing **L** which will toggle between inertial and non-inertial reference frames.

(a) Make sure k1 = k2 = 10, and that initial X is set to 0.25
(b) Press **F1**
(c) Press **P**, then steadily increase the value of Omega up to its critical value of 4.47. No need to pause the simulation between changes.
(d) How does the period of oscillation change with Omega?
(e) Use the expression for ‘oscillation frequency’ in the box to
 (i) Calculate the critical value of Omega
 (ii) Explain your answer to (d)
(f) Press **X** to disengage the experiment.

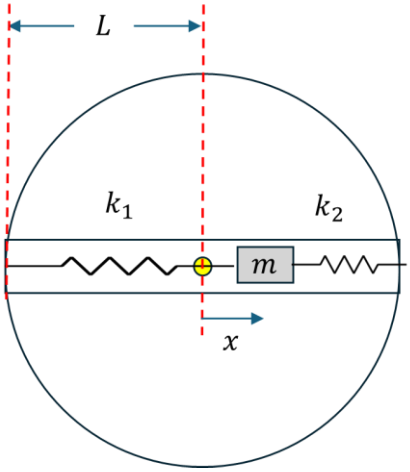
(3) Collecting and plotting data

Here you will essentially repeat (2) but also note down the oscillation period displayed on the HUD

(a) Make sure parameters are set to k1 = k2 = 10 and Omega = 0.
(b) Press **F1** to start the experiment, then repeat the following steps
 (i) Increase Omega a little
 (ii) Note down the period from the HUD
 (iii) Press **F2** then increase Omega a little.
(c) Press **X** when you are done, then enter your data into the Octave script **PeriodVsOmega.m** which you can find in the **Octave** folder.
(d) Run this script and your data will appear together with the period calculated from the expression in the box.

X.2. Investigation into different stiffnesses

Here we shall investigate the system when the values of k1 and k2 are not the same. If the mass starts off at the origin, then one spring



$$x(t) = [x(0) - C]\cos\omega t + C$$

Solution

$$\omega^2 = \frac{(k_1 + k_2)}{m} - \Omega^2$$

Oscillation frequency

$$C = \frac{L(k_2 - k_1)}{(k_1 + k_2) - m\Omega^2}$$

Offset from centre

$$m\ddot{x} = -(k_1 + k_2)x + L(k_2 - k_1) + m\Omega^2 x$$

Equation of motion

will exert a larger force, so we expect the equilibrium position of the spring to be offset from the origin. We need to investigate two things: (i) How rotation affects the offset, (ii) If the oscillation frequency differs from the ‘equal k1,k2’ case. The arrangement and revised key expressions are collected together in the box below.

(4) Offset as a function of rotation speed

Here we shall investigate how the rotation speed Ω affects the average distance of the mass from the centre of rotation, this is the ‘offset’ C.

(a) Set k1 to 9 and k2 to 11 (so the sum remains the same in the first experiments. Now set damping to 0.5. Make sure that Omega is set to 0. Set initial X to 0.1.

(b) Press **F1** to start the experiment then repeat the following steps

(i) Wait until the oscillations are damped

(ii) Note down the value of **meanX**.

(iii) Increase Omega a little (up to the critical value).

(c) Press **X** when you are done and enter your data into the Octave Script **DifferingKs1.m**

(d) Check your data agrees with the theoretical plot. Can you see how this is related to the expression given in the box above?

(e) Can you explain in simple and succinct English why the offset behaves as it does

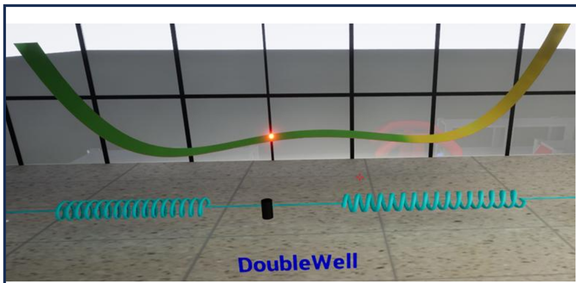
(5) Period as a function of rotation speed.

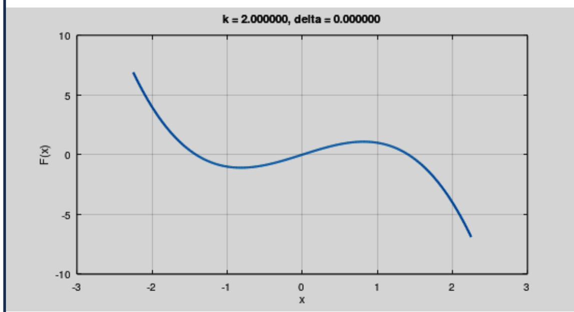
According to the expression in the box, the oscillation frequency does not change as the offset of the mass changes

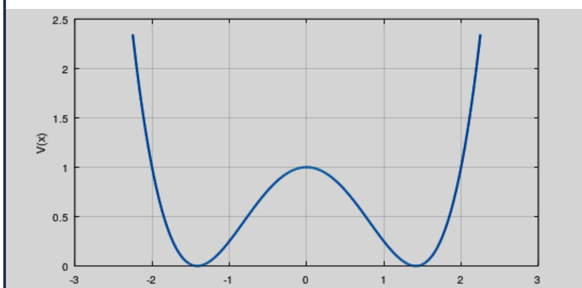
(a) Conduct a short investigation to verify this

(b) Can you explain in simple and succinct English why the oscillation frequency is independent of the offset?

X.3 Double Potential Well





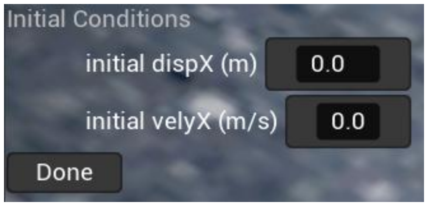


$$m\ddot{x} = kx - x^3$$

equation of motion

$$V(x) = \frac{1}{4}(x^2 - k)^2$$

potential



$$\omega = \sqrt{\frac{2k}{m}}$$

small amplitude oscilln. frequency

$$\omega = \sqrt{\frac{4k}{m} - \frac{3A^2}{m}}$$

within well oscillation frequency

$$\omega = \frac{1}{2}\sqrt{\frac{3A^2}{m} - \frac{4k}{m}}$$

across well oscillation frequency

X.3.1 Initial Experiments

The force and potential curves have been drawn for the default parameters. The key parameter here is k which fixes the potential minima at $x = \pm\sqrt{k}$ and the well 'edges' at $x = \pm\sqrt{2k}$ where we define the edge to be the locations where the potential is equal to its hump value. You will see from both force and potential curves that the origin is an unstable point.

How do you expect the system to behave if

(i) the bob starts close to the bottom of a potential well?
(ii) the bob starts close to the top of the potential hump?
(iii) the bob starts 'high up' on the potential curve?

(1) Exploring the Potential Wells

(a) Calculate the value of x at the potential minima and near the 'edges'. Press **I** and set **initial dispX** to locate the bob near the centre of a well.
(b) Press **F1** and observe the motion of the bob and its location on the hill.
(c) Leave the simulation running and press **I** and slowly increase the bob's starting location up to the well 'edge'. Note how the motion of the bob changes.
(d) Can you explain the change in bob's behaviour in simple and succinct English?

(2) Exploration outside of the Wells

(a) Continue increasing bob's starting location up to the largest value possible '**limit**' in the parameter menu.
(b) Again, can you explain your observations?
(c) Press **X** to deselect.

(3) Reading the solution trajectories
Here we shall work with Octave plots to get a better feeling of what is going on with the various trajectories.
(a) Reselect then set initial **dispX** to 1.6 so the bob is near the bottom of the well and press **F1** and run for 3 cycles or so. Press **F2**.
(b) Now set initial **dispX** to 1.999 so the bob is near the edge of the well and press **F1** to run for 3 cycles. Press **X** to exit.

(c) Press **O** to load up the Octave script **DoubleWell.m** and observe the two trajectory segments. How do the period and trajectory shape differ?
(d) Now look at the velocity segments. How do these differ between the segments?
(e) Can you use your answers to (d) to explain the difference in period and trajectory shapes?
(f) Press **X** to deselect.

X.3.2 Investigating Large Amplitude Solutions

Large amplitude solutions are very interesting to study. There exist theoretical results which describe these, see the appropriate chapter, but these, based on some form of energy balance involve approximations. In this section we shall collect some data, periods of large amplitude oscillations and see how well the approximate theory performs.

(4) Within-well solutions

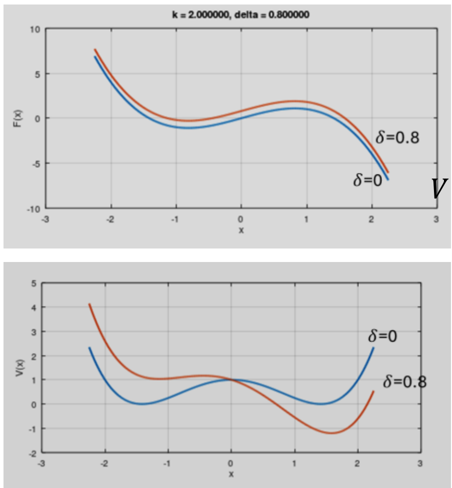
(a) To confine the solution within a single potential well, set **init DispX** to 0.1 and record the period from the HUD when it is stable.
(b) Repeat, increasing the initial displacement up to near the centre of the well which is at $\sqrt{k}$.
(c) Enter your data into the Octave script **InsideWell_Periods.m** and have a look at the plots[1]. Does the change in period with amplitude make sense to you?
(d) Comment on the accuracy of the theory.

(5) Across-Well solutions

(a) Set the **initial DispX** to 2.1 and record the period of oscillation.
(b) Progressively increase the displacement and record the periods. Decide yourself how large to make the displacement.
(c) Enter your data into the Octave script **AcrossWells_Periods.m** and have a look at the plots. The period dependence on amplitude is different from that discovered in (4). Does the different dependence make sense?
(d) Again, comment on the accuracy of the theory.

---

[1] There are two plots, one is the published He energy balance method. The second plot is ours where we have extended the He theory.



$$m\ddot{x} = kx - x^3 + \delta$$

equation of motion

$$V(x) = \frac{1}{4}(x^2 - k)^2 - \delta x$$

potential wells

$$\omega^2 = -\frac{(k - 3x_{Equ}^2)}{m}$$

small amplitude frequency

X.3.3 Deforming the Potential Well

The effect of the deforming ('unfolding') parameter δ is to change the relative depths of the potential wells, but also to shift the equilibrium location of the bottom of each well to the right. This will change the observed low-amplitude oscillation frequency. We are interested in finding out how δ affects this frequency

(6) Upper Branch Study

Let's look at the upper branch. The expression for frequency in the box shows that this depends on x_{Equ} the equilibrium position of the bob at the bottom of the well.

(a) Set the value of damping to 1.0 and initial x to 0.1. Press F1 and run the simulation so the bob comes to rest at the bottom of the well. Note down the equilibrium position

(b) Now progressively increase δ and note down the equilibrium position for each value. Press **X** when you are done.

(c) Insert your values into the Octave script **PeriodVsDelta.m**

Now let's make measurement of the oscillation period.

(d) Set damping to 0.0 and for each value of δ

(i) set initial x to the equilibrium value plus something small (e.g. 0.01) to get low-amplitude oscillations.

(ii) run the experiment and record the period from the HUD

(e) Now enter these measurements into the Octave script and run the script. Hopefully you will get good agreement with theory.

(7) Lower Branch Study

Repeat the above for the lower branch. To get the equilibrium positions start with initial $x = -0.1$ and apply damping. The rest of the experiment stays the same

(8) Understanding the Results

You should have found that increasing δ *decreases* the period for the upper branch and *increases* the period for the lower branch. Looking at the shape of the deformed potential well, can you suggest why this is, using simple and succinct English?