Chapter 3
Pendulums

3.1 A Brief Introduction

3.1.1 A Small dose of History

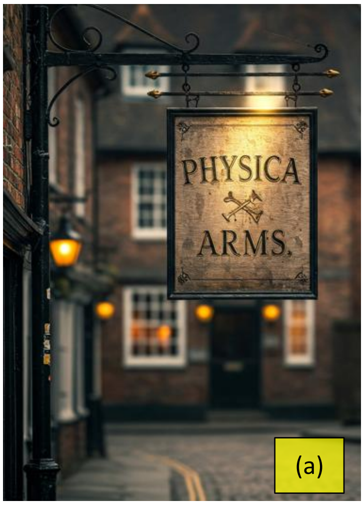
You have seen pendulums all around us, some typical examples intended to jog your memory are shown in Fig.1. The very first systematic pendulum observations were made around 1600 by Galileo who found that the period of the pendulum was a very significant property. He found this did not depend on the mass of the bob, nor on its amplitude, but was proportional to the square root of its length. Around 60 years later the Dutchman Christiaan Huygens took Galileo's work and invented the first pendulum clock. Clocks are very important things, they organize our lives, and they used to be fundamental in maritime navigation. Going backwards in history, the Romans used a pendulum-like 'hodometer' to measure distances, but not for clocks; they used sundials and water clocks, the latter were used in ancient Egypt as far back as the 16th century BC. The Chinese engineered an interesting application, Zhang Heng's seismometer from about 132 AD.

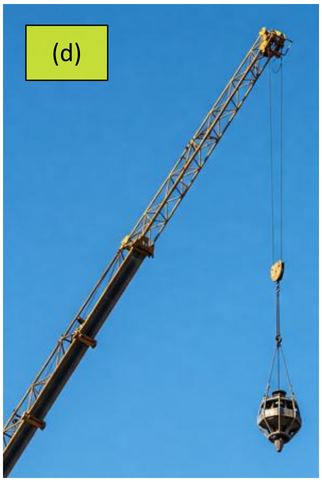
3.1.2 Ingredients to make a Pendulum

There are a few ingredients needed to successfully bake a pendulum. The first is *gravity*, without gravity you cannot have a pendulum. The second is some sort of *bob*, a mass that can move. Third there is a *pivot point* or some centre of rotation around which the bob can rotate and oscillate. Finally, the pendulum *system* needs to be engineered so that there is a rest or *equilibrium* position, and any push to move the bob away from this will result in the bob *returning to* this position. Without doubt, it is *gravity* which is most vital; if you create a working pendulum on Earth, then take it into deep space, where there is zero gravity, then it will stop being a pendulum, and perhaps become a work of art.

3.1.3 Pendulums in our Natural and Engineered World

Figure 1 shows a small collection pendulums which we think you will find interesting.





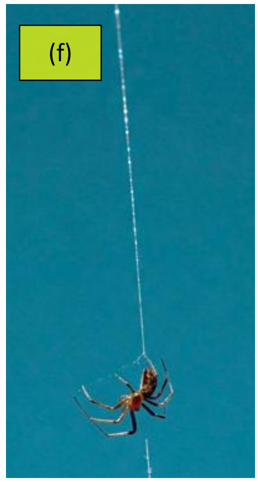
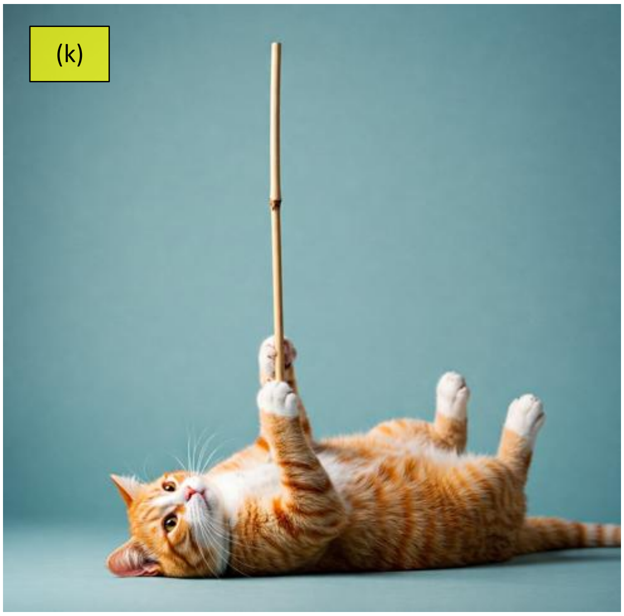


Figure 1

(a) Swinging pub sign
(b) Chandelier
(c) Pendulum on the Moon
(d) Wrecking ball
(e) Gremlin on a swing
(f) Spider hanging on a thread
(g) Swinging incense
(h) Construction crane with load
(i) Tree swaying in the wind
(j) Church Bell
(k) Cat balancing a pole
(l) Cat balanced on a pole



You will recognize some examples as ‘designed’ to oscillate such as the swing (e), the church bell (j), and the censer (g), and perhaps the wrecking ball (d). Others will oscillate even though they have not been designed to, such as the pub sign (a), chandelier (b) and the construction crane (h). In these cases oscillations may not be desirable or may be dangerous. The ‘natural’ examples (f) and (i) are interesting, the tree will oscillate when deflected by a strong gust of wind. The two cat examples (k), (l) are unusual since the pendulums here are ‘unstable’; without concerted action from the cat, the rod in (k) will topple and rotate onto the cat’s tummy, in (l) the cat will have to actively balance on the top of the rod, lest it falls. These are examples of ‘inverted’ pendulums.

3.2 The Simple Pendulum

The arrangement shown in Fig.2 is our starting point consisting of a mass in a gravitational field, constrained to rotate around a pivot point. Its equilibrium position is vertical with $\theta = 0$, and it is clear that if it is displaced by an increase in θ, so it appears as in the diagram, then it will move to reduce θ. We have all the ingredients of our recipe.

Figure 2.

The ODE for this system is straightforward if we choose to work in polar coordinates: Our ‘displacement’ variable is θ and we shall have an angular velocity $\dot{\theta}$ and an angular acceleration $\ddot{\theta}$ which we shall need to set up the ODE. Instead of force we need to use torque τ (force times distance from the centre of rotation, the blue line in Fig.??), and instead of mass we need moment of inertia, which here is mL^2. The torque on the bob is just

$$\tau = -mgL\sin\theta,$$

so the ODE is

$$mL^2\ddot{\theta} = -mgL\sin\theta,$$

hence

$$\ddot{\theta} = -\frac{g}{L}\sin\theta. \qquad (1)$$

Note that the bob mass has disappeared from the expression[1] and the pendulum behaviour is determined by gravity and the pendulum length. Remember Galileo found the length was significant by experiment.

Equation 1 looks so benign, even friendly that we expect to find a simple analytical solution. But mathematics intervenes, and there is no solution to this equation. So we must resort to investigating approximations, and that requires some thought, which of course makes life interesting. Do not fret – we have done the groundwork in Chapter 2.

3.2.1 The 'classic' textbook approximation.

The problem with eq.1 is that the right hand side has a non-linear function of our system variable, $\sin\theta$. We really want a linear function in terms of θ and we know that the Taylor expansion of $\sin\theta$ provides us with that. But let's step beyond the classic textbook discussion.

The Taylor expansion of $\sin\theta$ around zero is just

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots \qquad (2)$$

so if we take the first term as an approximation, the ODE becomes

$$\ddot{\theta} = -\frac{g}{L}\theta. \qquad (3)$$

We recognize this immediately, and can write down the expression for the oscillation frequency,

$$\omega = \sqrt{\frac{g}{L}} \qquad (4)$$

We can of course take the next term in the approximation and attempt to solve the ode

[1] You should not be surprised by this, since the physics involves inertial mass and gravitational mass, which since they are equal cancel from the acceleration.

$$\ddot{\theta} = -\frac{g}{L}\left[\theta - \frac{1}{3!}\theta^3\right] \qquad (5)$$

We proceed as usual an try a solution $\theta = A\cos\omega t$ which on substitution into eq.?? gives us

$$-\omega^2 A\cos\omega t = -\frac{g}{L}\left[A\cos\omega t - \frac{A^3}{6}\cos^3\omega t\right]$$

$$= -\frac{g}{L}\left[A\cos\omega t - \frac{A^3}{6}\left(\tfrac{1}{4}\cos 3\omega t + \tfrac{3}{4}\cos\omega t\right)\right] \qquad (6)$$

We choose to keep only the terms in ωt and neglect the third harmonic which gives us

$$\omega^2 = \frac{g}{L}\left[1 - \frac{A^2}{8}\right] \qquad (7)$$

or for amplitudes not too large,

$$\omega \approx \sqrt{\frac{g}{L}}\left(1 - \frac{A^2}{16}\right) \qquad (8)$$

It's interesting to see just how good the series expansions for the sin function are. Here's a short table of percent errors, calculated for an angle 45 deg.

$\sin\theta$	0%
θ	11%
$\theta - \frac{1}{3!}\theta^3$	0.35%
$\theta - \frac{1}{3!}\theta^3 + \frac{1}{5}\theta^5$	0.005%

3.2.2 The Stiffness Averaging Approach

Here we apply the results derived in Chapter 2.2.2. There we found how the oscillation frequency was related to the partial derivative of force with displacement, and took an average of this derivative over the displacement from 0 up to the amplitude *A*. We repeat that analysis here for the simple pendulum.

Our ODE is

$$\ddot{\theta} = \frac{g}{L}F(\theta), \qquad F(\theta) = \theta - \frac{\theta^3}{6}, \qquad (9)$$

and the partial derivative $\partial F/\partial\theta$ is just

$$\frac{\partial F}{\partial\theta} = 1 - \frac{\theta^2}{2}$$

Applying eq.?? from Chapter 2 and simplifying we find

$$\omega(A) = \sqrt{\frac{g}{L}}\sqrt{\left(1 - \frac{7}{48}A^2\right)}\ . \qquad (10)$$

We shall return to discuss this soon but first let's take a more direct approach. As explained in chapter 2, the simplest 'averaging' approximation for the frequency uses the partial derivative at the half-amplitude mark $F(\theta = A/2)$. So there is no need to do the series expansion, since we have an analytic form $F(\theta) = (g/L)\sin\theta$. We proceed directly,

$$\omega = \sqrt{\frac{g}{L}}\sqrt{\left.\frac{\partial F(\theta)}{\partial\theta}\right|_{A/2}} = \sqrt{\frac{g}{L}}\sqrt{\cos(A/2)}\ . \qquad (11)$$

3.2.3 The Energy Approach

Here we apply the approach introduced in Section X.X.X where the period is calculated from the energy integral, evaluated over a quarter-period of a single oscillation of amplitude A the result multiplied by 4 to get the total period

$$T = 4\sqrt{\frac{L}{g}}\int_0^A \sqrt{\frac{1}{2(E - V(\theta))}}\,d\theta \qquad (12)$$

Let's take our time to understand how to use the integral, figure 3 will provide some guidance. The pendulum is shown in three positions; vertically at rest, at its largest angle A, where it is also at rest, and intermediate angles θ where it is moving with angular velocity $\dot{\theta}$. At the amplitude position, the total energy E_A is wholly potential,

$$E_A = mgL(1 - \cos A)\ , \qquad (13)$$

and at angles between 0 and A the potential energy is

$$V(\theta) = mgL(1 - \cos\theta). \qquad (14)$$

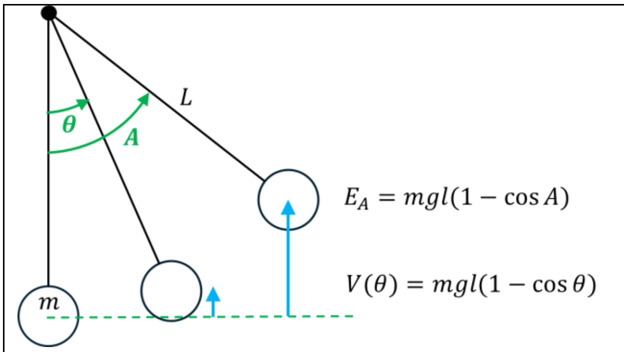


Figure 3.

Substitution eqs.13 and 14 into eq.12 gives the required expression for the period,

$$T = 4\sqrt{\frac{L}{2g}}\int_0^A \sqrt{\frac{1}{(\cos\theta - \cos A)}}\,d\theta \qquad (15)$$

How have we chosen the integration limits? We need to cover the entire angle range experienced by the oscillating pendulum; this is just from 0 to *A*. To better understand the meaning of this calculation, let's tave a look at the integral

$$I(A) = \int_0^A \sqrt{\frac{1}{(\cos\theta - \cos A)}}\,d\theta \qquad (16)$$

which is plotted in Fig.4. It's value as $A \to 0$ is just $\pi/\sqrt{2}$ and it rises sharply as $A \to \pi$. Now to make things a little clearer, we can re-write eq.15 using $I(A)$ and expressing the period in a more familiar form, the small angle approximation from the ODE analysis,

$$T(A) = 2\pi\sqrt{\frac{L}{g}}\frac{\sqrt{2}}{\pi}I(A). \qquad (17)$$

We see that as $A \to 0$ we recover the small angle approximation for the period exactly.

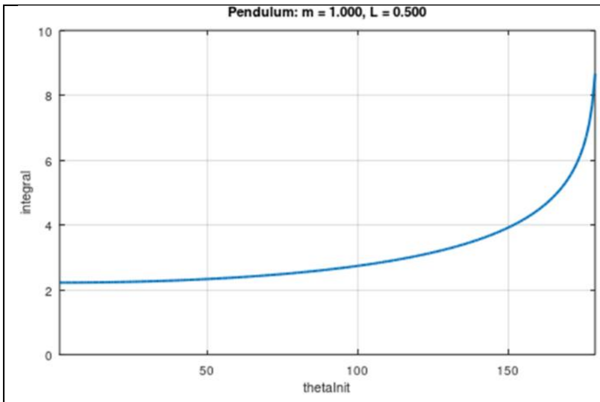


Figure 4.

A comparison of the period from eq.17 with results from accurate numerical solution of the ODE is shown in Fig.5. Note that agreement is very good and especially that this is maintained for pendulum initial angles (amplitude) of greater than 90 degs, which of course corresponds to an *inverted pendulum*. The low-angle period of $T = 2\pi\sqrt{L/g}$ = 1.4185036 s is also in agreement.

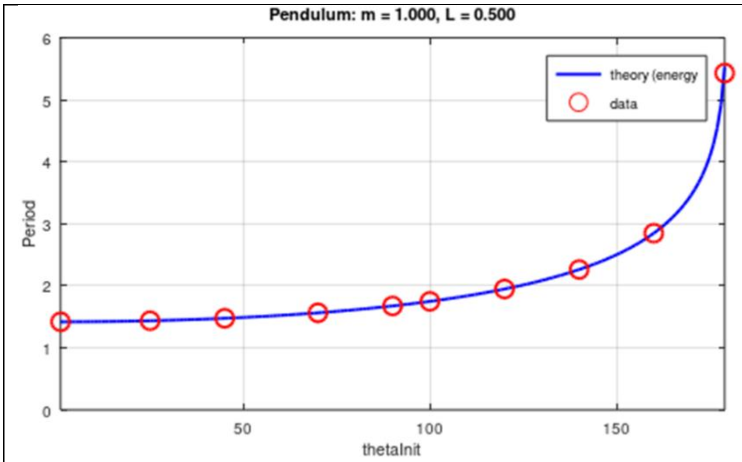


Figure 5.

3.2.4 He's Energy Balance Approach

Here we apply the method we used in Section.X.X where we used an approximation written as a power series of the state variable. This makes sense, since He's approach results in a polynomial equation for the amplitude *A*.

Starting with the expression for energy

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta)$$

we expand $\cos\theta$, and assuming $\theta(0) = A$ and $\dot{\theta}(0) = 0$ the energy balance equation leads to

$$\dot{\theta}^2 + \frac{g}{L}\left[\theta^2 - \frac{\theta^4}{12} + \frac{\theta^6}{720}\right] = \frac{g}{L}\left[A^2 - \frac{A^4}{12} + \frac{A^6}{720}\right]. \quad (18)$$

The residual function becomes

$$R(t) = \dot{\theta}^2 + \frac{g}{L}\left[\theta^2 - \frac{\theta^4}{12} + \frac{\theta^6}{720} - A^2 + \frac{A^4}{12} - \frac{A^6}{720}\right]$$

Substituting $\theta = A\cos\omega t$, this becomes

$$R(t) = \omega^2 A^2 \sin^2\omega t + \frac{g}{L}\left[A^2\cos^2\omega t - \frac{A^4}{12}\cos^4\omega t + \frac{A^6}{720}\cos^6\omega t - A^2 + \frac{A^4}{12} - \frac{A^6}{720}\right] \quad (19)$$

and co-locating at $\omega t = \pi/4$ we end up with the result

$$\omega^2 = \frac{g}{L}\left[\theta^2 - \frac{1}{8}\theta^4 + \frac{7}{2880}\theta^6\right] \quad (20)$$

3.2.5 Forces, Potentials and Phase Planes

Now we have derived expressions for approximations to forces and potentials, we are able to plot some of these out. Perhaps the most informative are the potentials, Fig.6. The exact potential, $E = mgL(1 - \cos\theta)$ is plotted in red clearly showing its periodic nature. The first approximation, $E = mgL(\theta^2/2)$ is plotted in green, and agrees well close to $\theta = 0$. The second approximation $E = mgL(\theta^2/2 - \theta^2/24)$ is plotted in blue and shows an improved fit near $\theta = 0$. Of course neither approximations can match the actual periodic function.

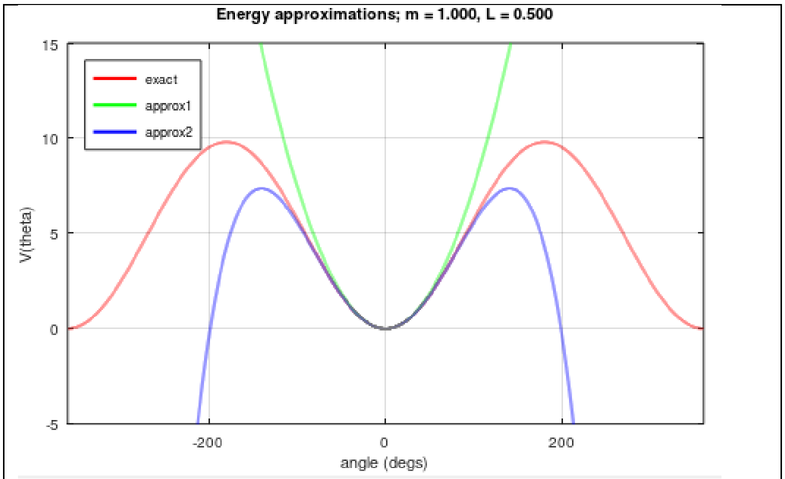


Figure 6.

The force approximations are shown in Fig.7 where the first approximation $F = -mgL\theta$ and the second approximation $F = -mgL(\theta - \theta^3/6)$ are expressed as ratios of the exact force $F = -mgL\sin\theta$. The plot therefore indicates where the agreement is 10% or better. The lowest-order approximation has an error of around 2.5% at 90 degs while the higher-order approximation does better at around 0.5%. These values may be useful in planning laboratory activities.

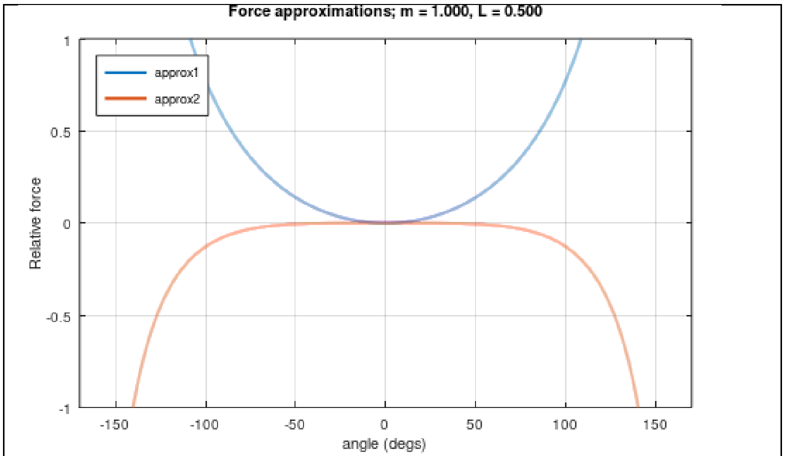


Figure 7.

The phase diagrams are perhaps the most informative, since these capture the most information about the system. The phase diagram for the exact system is shown in Fig.8 which displays a series of localized stable solutions around the equilibrium points 0, $2\pi, 4\pi, \ldots$. These orbits are stable for low amplitudes, below 180 deg, anything larger results in rotation of the pendulum around its centre, where it passes periodically 'over the top. The exact trajectory depends on the initial velocity of the bob, at large velocities the angular displacement shows less 'wobble', as the effects of the restoring force become smaller, and the bob assumes more of a steady rotating motion.

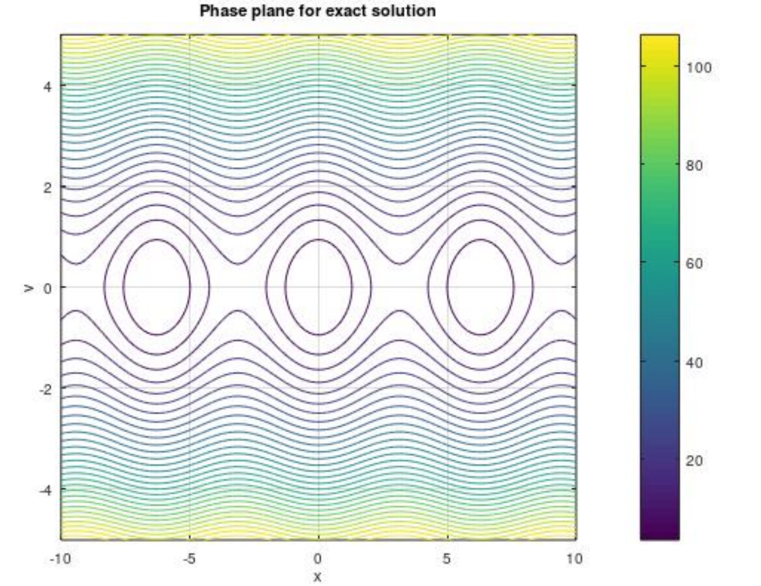


Figure 8.

The two approximations to this are shown in Fig.9. As expected neither show any periodicity; the lower order approximation has a series of concentric ellipses with no sign of deformation for larger amplitudes. Such a phase diagram is typical for linear systems. The higher approximation begins to show some 'tails' and reveals some of the nonlinearity in the system.

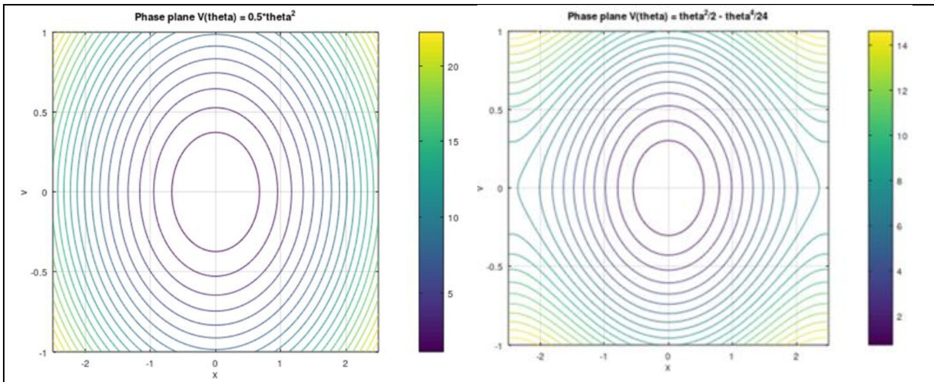


Figure 9.

3.2.6 Roundup, Comparisons and Discussion

Now we can evaluate the accuracy of our various approximations against the accurate numerical solution of the ODE. Table.1 presents a few solutions and Fig.10 shows a plot of solutions versus amplitude.

A(deg)	T_{soln}	T_{energy} (eq.??)	T_{stiff} (eq.??)	$T_{HeEnergy}$ (eq.??)
1	1.418530	1.418530	1.418530	1.418530
45	1.475206	1.475206	1.475784	1.475832
90	1.674317	1.674317	1.686894	1.687771
120	1.947330	1.947436	2.006067	2.009172
160	2.847034	2.847656	3.404045	3.410108

Table 1.

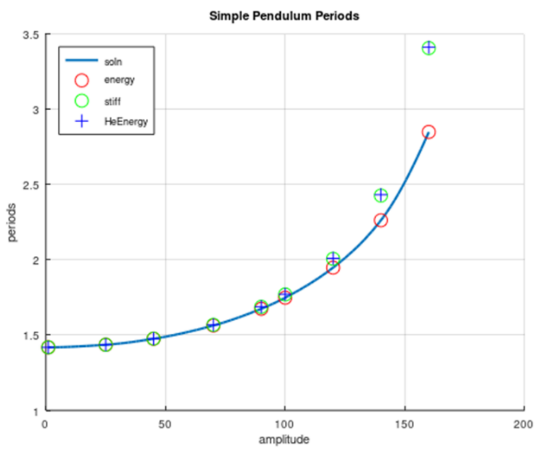


Figure 10.

Clearly the energy approximation is the best and is highly accurate. This is unsurprising since it does not involve any power series expansion, truncation, and therefore explicit approximation. The energy approximation is as good as the numerical integration procedure.

Nevertheless, we suggest that the stiffness approach, eq.11, and He’s energy approach, eq.20, are also of great use. They provide decent approximations up to around 90 degs, but their strength lies in the fact that they provide an analytical solution – a ‘formula’ – which allows us to see at once how the amplitude affects frequency and period. In this case, a larger amplitude decreases the frequency and increases the period. Also the formulae give us a feel of what magnitude of amplitude will lead to a substantial deviation from the zero-amplitude case. For example in eq.?? we could estimate the amplitude which makes the term $(7/48)A^2$ substantially less than 1. A simple calculation suggests this is around 65 degs.

3.3 The Dumbbell Pendulum

3.3.1 System Dynamics

This is a development of the simple pendulum where we add a second mass and connect the masses via a rod whose mass we ignore. The dumbbell is arranged to rotate in the vertical plane around a point located somewhere along the rod. The location of this point will be a system parameter a. The scenario is shown in Fig.11 with forces on the masses indicated.

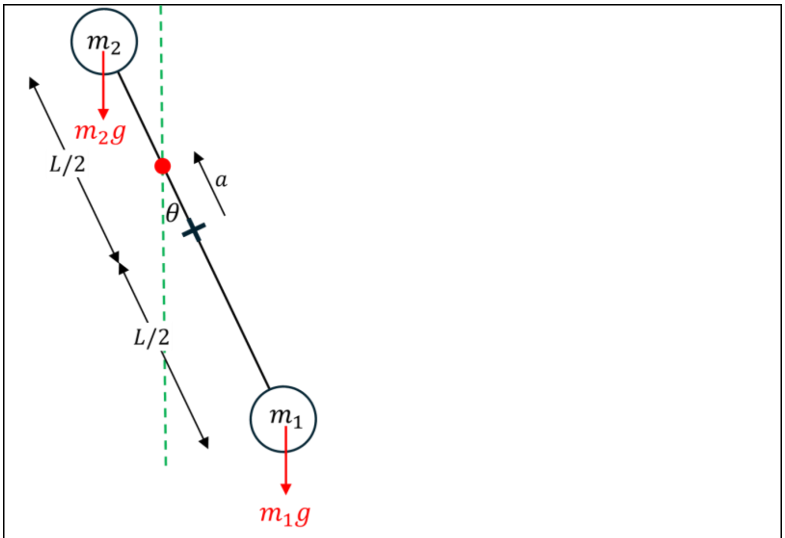


Figure 11. Cross shows mid-point, not cofm.

The rotational dynamics of this system is governed by the ODE, where we neglect damping,

$$I\ddot{\theta} = \tau, \qquad (21)$$

where the torque (positive in a clockwise direction) is

$$\tau = [m_2(L/2 - a) - m_1(L/2 + a)]g \sin\theta \qquad (22)$$

and the moment of inertia around the pivot point is

$$I = m_1(L/2 + a)^2 + m_2(L/2 - a)^2 \qquad (23)$$

and substituting into eq.21 we have

$$\ddot{\theta} = -\frac{m_1(L/2+a) - m_2(L/2-a)}{m_1(L/2+a)^2 + m_2(L/2-a)^2}\, g \sin\theta. \qquad (24)$$

We can reduce this to a familiar equation for the simple pendulum

$$\ddot{\theta} = -\frac{g}{L'} \sin\theta$$

where we have defined the *equivalent length*

$$L' = \frac{m_1(L/2+a)^2 + m_2(L/2-a)^2}{m_1(L/2+a) - m_2(L/2-a)} \qquad (25)$$

which means we can effectively replace the dumbbell with a simple pendulum of this equivalent length.

Eq.25 is rather cumbersome, so we shall set $m_2 = m_1$ and continue with this simplification unless we state otherwise. Then the expression for equivalent length becomes much more transparent

$$L' = \frac{L^2}{4a} + a \qquad (26)$$

We shall also assume, (at least for the moment), that $a > 0$ therefore $L' > 0$.

3.3.2 Equivalent length and small-angle oscillations

Using eq.26 we can immediately write down the expression for the frequency of small-angle oscillations

$$\omega_0^2 = \frac{g}{\left(\frac{L^2}{4a} + a\right)} \qquad (27)$$

Let's look at a few examples with various values of a and L' and their oscillation frequencies, Fig.12.

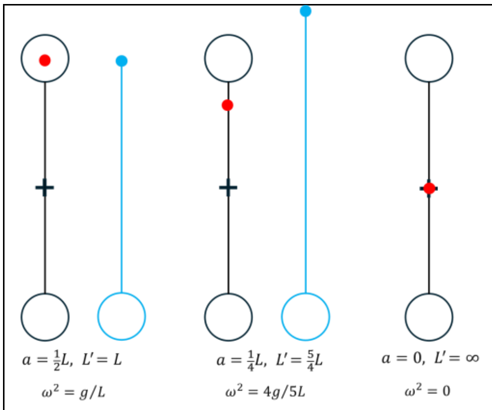


Figure 12.

Starting with the pivot point at the centre of the top mass, we have of course the simple pendulum. Progressively lowering the pivot we find that L' steadily increases and the oscillation frequency decreases. When we hit the mid-point between the masses, $L' = \infty$ and the oscillation frequency is zero, not surprising since there is no net torque on the system.

A plot of effective length versus a is shown in Fig.13 showing the asymptote as $a \to 0$. Remember that this is drawn for $m_2 = m_1$.

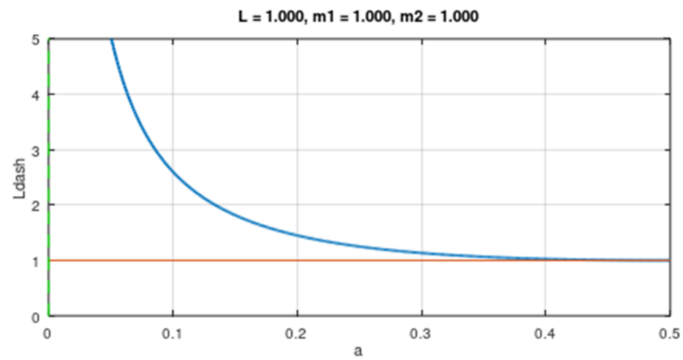


Figure 13.

Of course you can argue for the frequency variation with pivot point location by direct use of the two torques on the system. As the pivot point is raised, the torque on the upper mass decreases and the torque on the lower mass increases. So there is a net increase in the restoring torque which leads to an increase in oscillation frequency.

3.3.2 Pivot point located below the mid-point

When the pivot point is located below the mid-point, we have effectively an inverted pendulum. All the above analysis must apply, we have $a < 0$ and it follows that $L' < 0$, so we must take care in applying eq.27 to calculate the oscillation frequency. But there is another way we could proceed which does not require a negative effective length.

We could think of the equivalent situation, where the system is rotated 180 deg so that the pivot point finds itself above the mid-point; of course we need to change the initial displacement angle. The transformation we are thinking about is shown in Fig 14. The original situation is shown in (a) where the experimental configuration is shown together with the equivalent pendulum in blue. The initial angle is θ_{init}. The angle of oscillation is indicated here by the blue arc. The target situation is shown in (b) and in (c) we show the original situation rotated. Remember this situation has the pivot point located above the mind-point but has an initial angle of $\theta_{init} + \pi$, so it looks the same as the original situation.

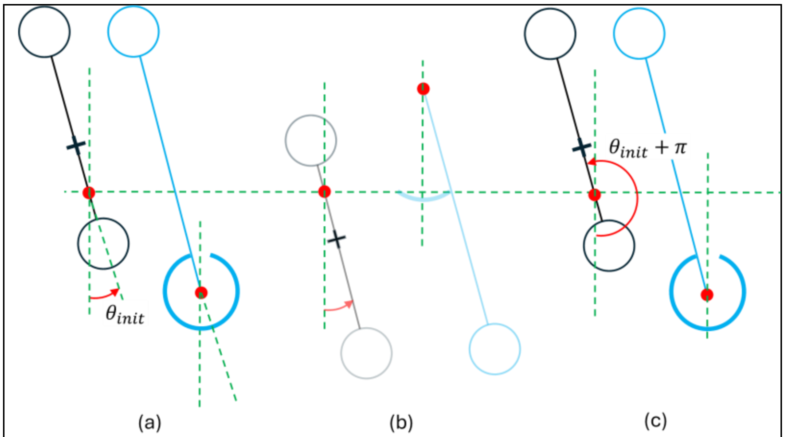


Figure 14.

3.3.3 Large Amplitude Period calculations

The energy method is of course ideal for making these calculations. We repeat the expressions required where we are using the effective length L' and we will use the above transformation when needed to make this positive.

$$T(A) = 2\pi \sqrt{\frac{L'}{g} \frac{\sqrt{2}}{\pi}} I(A). \qquad (28)$$

We must pay attention to the limits in the integral; for our ‘standard’ system we have

$$I(\theta_{init}) = \int_0^{\theta_{init}} \sqrt{\frac{1}{(\cos\theta - \cos\theta_{init})}} d\theta, \qquad (29)$$

and for our transformed system we have

$$I(\theta_{init}) = \int_{\theta_{init}}^{\theta_{init}+\pi} \sqrt{\frac{1}{(\cos\theta - \cos(\theta_{init} + \pi))}} d\theta. \qquad (30)$$

A plot of period versus a is shown in Fig.15. Note that the plot is labelled with a in the range $-L/2 \ldots L/2$; this is only for clarity, we do not use negative values of a!

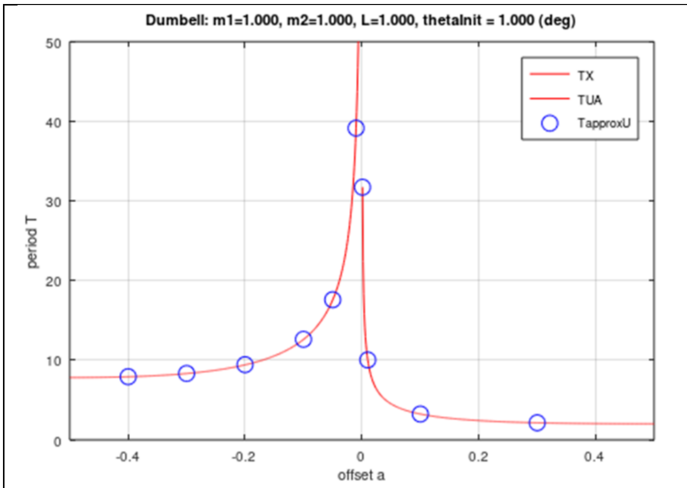
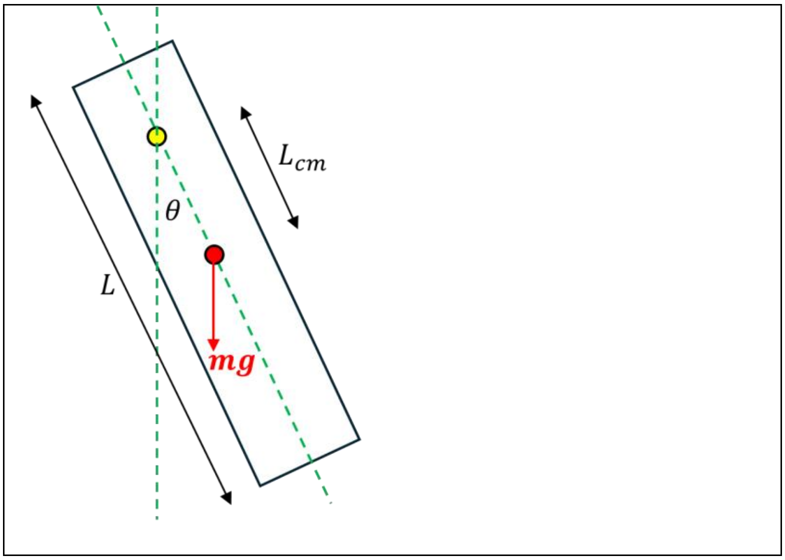


Figure 15.

3.4 The Physical Pendulum

This is a standard textbook example, where the pendulum’s mass is distributed throughout a body which is then pivoted, and its oscillations depend on the location of the pivot point. We shall take a specific example, a uniform bar of length Lshown in Fig.??.. The pivot point is located at the yellow point which is located above the centre of mass of the bar shown by the red point. As usual we must consider the torque applied by gravity around the pivot point and need to know the moment of inertia of the bar around this point.



When displaced an angle θ as shown, the restoring torque due to the weight of the bar is just

$$\tau = -mgL_{cm}\sin\theta \qquad (xx)$$

where L_{cm} is the distance form pivot point to the centre of mass. Now the moment of inertia for the bar about its centre of mass turns out to be $(1/12)mL^2$ so using the parallel axis theorem, the moment of inertia about the pivot point becomes

$$I_P = \frac{1}{12}mL^2 + mL_{cm}^2 \qquad (xx)$$

which is seen to increase as the offset of the pivot point L_{cm} increases. This will have an effect on the frequency of oscillation as we shall see. The dynamics of the system is governed by the usual ODE

$$I_P\ddot{\theta} = \tau$$

i.e.,

$$\left(\frac{1}{12}mL^2 + mL_{cm}^2\right)\ddot{\theta} = -mgL_{cm}\sin\theta \qquad (xx)$$

which again reduces to the familiar simple pendulum expression

$$\ddot{\theta} = -\frac{g}{L'}\sin\theta$$

where the equivalent length L' is simply

$$L' = \frac{L^2}{12L_{cm}} + L_{cm} \qquad (xx)$$

which reminds us of eq.?? for the dumbbell pendulum. Applying the small angle approximation $\sin\theta \approx \theta$ we obtain the oscillation frequency

$$\omega^2 = \frac{g}{\left(\frac{L^2}{12L_{cm}} + L_{cm}\right)} \qquad (xx)$$

which of course resembles eq.?? This is interesting approaches since it shows the dependence of the frequency on L_{cm}; due to the first term in the denominator, the frequency will approach zero as L_{cm} approaches zero, i.e. as the pivot point approaches the centre of mass.

You might find this result a little surprising, since looking at it slightly differently, as the length of the part of the bar below the pivot point increases, so does the oscillation frequency. This is clearly not the same as in the case of the simple pendulum.