Chapter 2
Nonlinear Oscillations

2.1 A Brief Introduction

Many physical oscillators or oscillators applied in engineering are non-linear; this means the force-extension curve is non-linear. Instead of the ODE describing a linear system, such as a mass on a spring where the force is clearly linear,

$$m\ddot{x} = g(x), \qquad g(x) = -kx, \qquad (1)$$

we have the more general system

$$m\ddot{x} = f(x), \qquad (2)$$

Where $f(x)$ is not simply proportional to x but may contain powers of x or more exotic functions such as trig functions. We shall start by considering an important function which appears in the Duffing oscillator,

$$m\ddot{x} = -kx - \alpha x^3 \qquad (3)$$

Before we get into details, let's have a look at some basic facts about this equation, such as where it is applied. Many systems comprising elastic elements can be modelled using simple linear springs, but often when the displacement becomes large this simple model is inadequate. A well-known example is the cantilever beam, Fig.1 with one end fixed to a support and the other connected to a load.

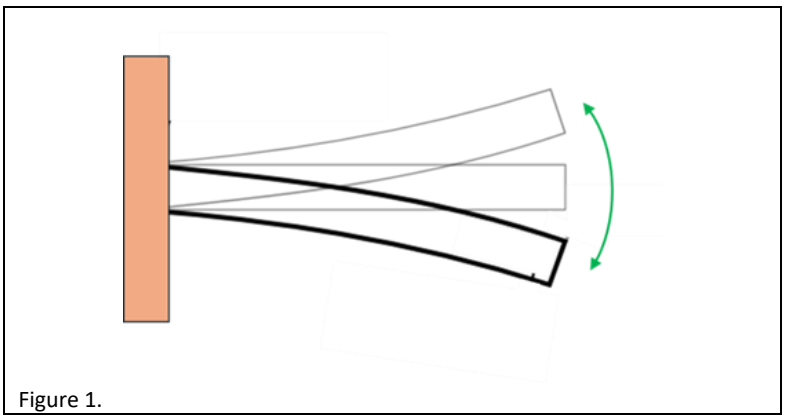


Figure 1.

A mass placed on this beam will obey eq.3 including large deflexions. Note that the beam is symmetrical in a vertical sense, the restoring force is the same for an upward and equivalent downward deflexion. This is reflected in the cubic power in eq.3, a quadratic power would be incorrect; it does not have the correct symmetry.

Graphs of force $f(x) = -kx - \alpha x^3$ and potential energy $V(x) = \frac{1}{2}kx^2 + \frac{1}{4}\alpha x^4$ are shown together in Fig.2

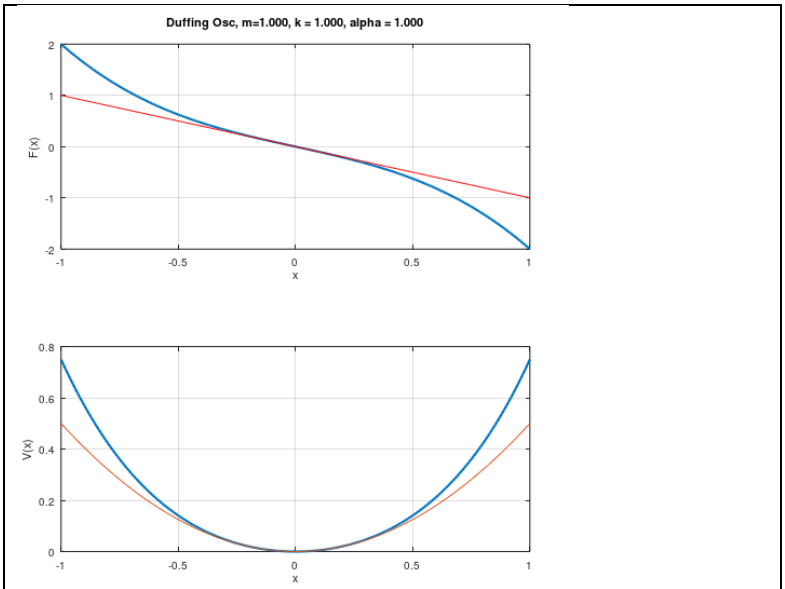


Figure 2.

which are again symmetrical about the equilibrium position $x = 0$. The curves for the linear case $f(x) = -kx$ are drawn in red. These curves are simple, e.g., the potential curve is symmetric and only has a single minimum. Close to the minimum the linear approximation looks quite good.

Finally we consider the phase plane for eq.3, a plot of $\dot{x}$ versus x. If you are unsure about the meaning of the phase plane and how to construct it, review Chapter X.X. In this case we have a series of concentric ellipses, each corresponds to a pair of initial conditions $x(0), \dot{x}(0)$ in other words to a different amplitude of oscillation. All trajectories are closed, so each particular oscillation continues with the same amplitude through time; trajectories are smooth ellipses which is a hallmark of *linear* oscillations.

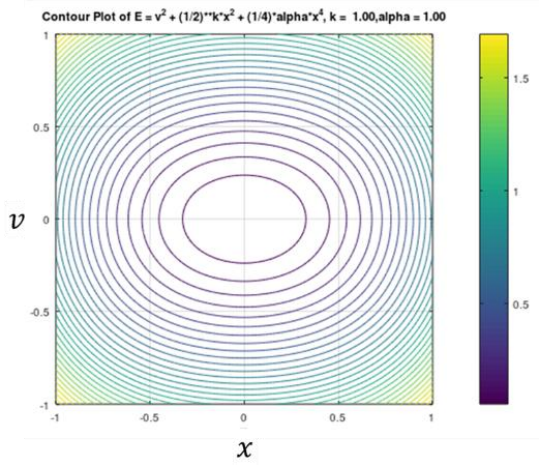


Figure 3.

2.2 The Duffing Equation

2.2.1 Variable Stiffness defines Variable Frequency

The form of the force $f(x) = -kx - \alpha x^3$ is symmetric around the origin, since it only contains odd powers of *x*. We can easily find the locations where the force is zero by solving $-kx - \alpha x^3 = 0$ which has a single solution $x = 0$.

We now wish to find an expression for the frequency of oscillation around this point. Let's start by reminding ourselves of the argument for a linear oscillator where we start with the ODE

$$m\ddot{x} = F(x) = -kx,$$

And try a solution of the form $x(t) = A\cos\omega t$ which gives following substitution into the ODE

$$-m\omega^2 = -k, \quad \text{hence} \quad \omega = \sqrt{k/m} \qquad (4)$$

which is very familiar. Let's focus on the physical meaning of *k* the spring constant or 'stiffness'. This is the force per extension which we can write formally as

$$-k = \frac{\Delta F}{\Delta x} = \frac{\partial \mathrm{F}}{\partial \mathrm{x}}.$$

These are equal since the force is proportional to displacement in this case. Now let's replace *k* in eq.4 with the partial derivative to give us

$$\omega = \sqrt{-\frac{1}{m}\frac{\partial \mathrm{F}}{\partial \mathrm{x}}}\,. \qquad (5)$$

The insight here, is that the frequency of oscillation is related to the gradient of the force-extension graph. Of course you knew that already, but it's nice to have it stated in maths. We shall now make use of this expression in discussing the frequencies of non-linear oscillations.

2.2.2 Application to the Duffing Oscillator

Let's take our Duffing oscillator, eq.3, and move it away from the origin of the force-extension curve so it oscillates, hypothetically around a new location x_0. Note this may require some magic, since we do not want to interfere with the force-extension function. Figure 4 illustrates what we have in mind; the mass shown to a distance *x* from the origin together with the forces at these points.

We know it is the gradient of the force-extension curve, here near x_0 that determines the oscillation frequency, which from the figure is just

$$\frac{F(x) - F(x_0)}{x - x_0} = \left.\frac{\partial F}{\partial x}\right|_{x_0}, \qquad (6)$$

where the gradient is evaluated at x_0. You may be thinking that

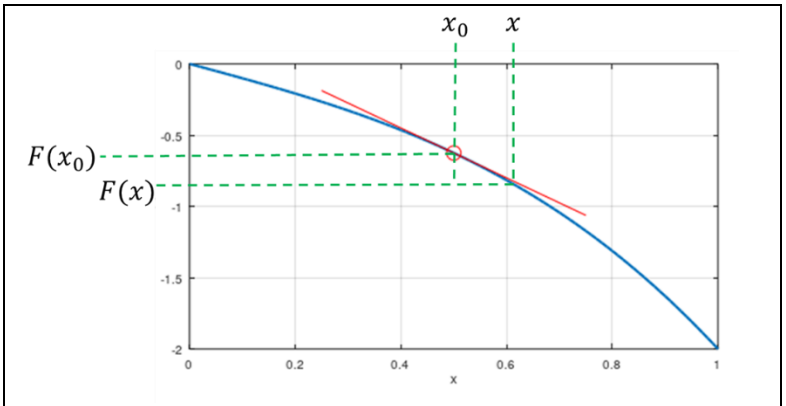


Figure 4.

this looks as though it may come from the Taylor expansion of the force, and you are right; here it is,

$$F(x) = F(x_0) + \left.\frac{\partial F}{\partial x}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{\partial^2 F}{\partial x^2}\right|_{x_0}(x - x_0)^2 + \cdots \quad (7)$$

So now we can generalize eq.5 for any point x_0 on the non-linear force-extension curve,

$$\omega = \sqrt{-\frac{1}{m}\frac{\partial F}{\partial x}\Big|_{x_0}}. \qquad (8)$$

For our Duffing equation this becomes

$$\omega(x_0) = \sqrt{\frac{k}{m} + \frac{3\alpha x_0^2}{m}}. \qquad (9)$$

We see something nice in this expression; the first term inside the root is just our linear term, the second term adds a correction to the frequency due to the cubic force term, and this acts to *increase* the oscillation frequency. We can understand this, since the stiffness of the spring increases with x_0.

However, eq.9 has *very limited applicability* in fact it is almost useless, since the frequency it predicts is constantly changing with x_0 as the mass is oscillating! What we need is to take some sort of *average*, and we shall now turn to doing this.

2.2.3 Frequency from the Averaged force gradient

Let's start with oscillations with an amplitude A and ask how to apply eq.8 to make a sensible approximation of the frequency. The situation is shown in Fig.5. Taking forward the idea of using an average, it makes sense to calculate the frequency at the mid-range oscillator motion interval $[0 - A]$, i.e., $A/2$.

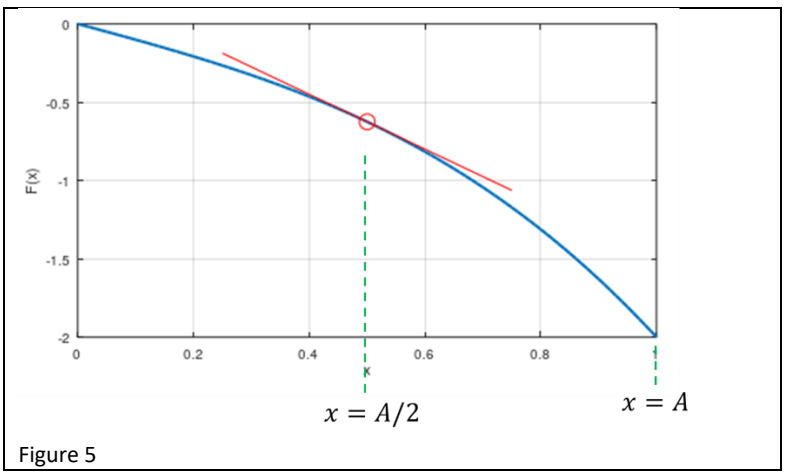


Figure 5

Applying this to the Duffing force expression gives us

$$\omega(A) = \sqrt{\frac{k}{m} + \frac{\alpha}{m}\frac{3}{4}A^2}\,. \qquad (10)$$

Now, unlike eq.9 this expression *is very useful* since it predicts how the oscillation frequency depends on amplitude. At this point it is interesting to compare its predictions, with results obtained by accurate numerical solutions of the ODE, eq.10, look forward to Table.1. The results shown there are amazing, for each amplitude the agreement is excellent, but the most amazing result is that the approximation works over a *huge range* of amplitudes.

2.2.4 Generalizing the Approach

Rather than taking a single point midway through the amplitude range, we could divide the amplitude range into$(N-1)$ intervals, calculate ω^2 for each interval and take the average of these. You can see the idea in Fig.6 where we have chosen three intervals.

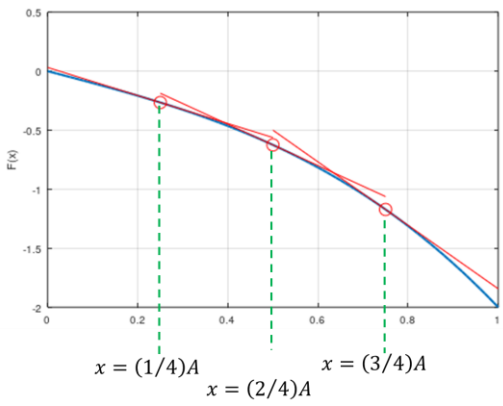


Figure 6.

We must extend eq.9, remembering to divide by 3 to get the average.

$$\omega = \sqrt{-\frac{1}{3m}\left(\left.\frac{\partial F}{\partial x}\right|_{A/4} + \left.\frac{\partial F}{\partial x}\right|_{A/2} + \left.\frac{\partial F}{\partial x}\right|_{3A/4}\right)} \qquad (11)$$

Applying this to our Duffing oscillator, we find,

$$\omega(A) = \sqrt{\frac{1}{3m}\left(k + \frac{3}{16}\alpha A^2 + k + \frac{3}{4}\alpha A^2 + k + \frac{27}{16}\alpha A^2\right)},$$

which simplifies to

$$\omega(A) = \sqrt{\frac{k}{m} + \frac{\alpha}{m}\frac{7}{8}A^2}\,. \qquad (12)$$

If you would like an expression for the general case, then here it is

$$\omega^2 = \frac{1}{N-1}\sum_{i=1}^{N-1}\omega_i^2, \qquad \omega_i^2 = -\frac{1}{m}\frac{\partial F}{\partial x}\bigg|_{x_i} \qquad (13)$$

2.2.3 Limitations of this approach

The kernel of the approach described above is to calculate the average of the derivative $\partial F/\partial x$ over the interval experienced by the oscillator, which we have taken as the amplitude of oscillation. This implies that the amplitude is well-defined in limiting this interval. For the Duffing equation where the nonlinearity is symmetric in *x* then the oscillator will move between $-A$ and $+A$ and so we can use the amplitude in calculating the approximate frequency. Things aren't so straightforward if the nonlinearity is not symmetric, e.g., if it includes even powers of *x*. We shall discuss this in detail in Section.X.X.

2.2.4 The Energy Approach

We have a conservative system since there is no damping. We can express the total constant energy *E* as a sum of kinetic energy and potential energy terms,

$$\tfrac{1}{2}m\dot{x}^2 + V(x) = E\,, \qquad (14)$$

from which we have

$$\dot{x} = \sqrt{\frac{2(E-V(x))}{m}}, \quad \text{so} \quad \frac{dt}{dx} = \sqrt{\frac{m}{2(E-V(x))}} \qquad (15)$$

which is integrated

$$t = \int_0^t dt' = \int_0^x \sqrt{\frac{m}{2(E-V(x'))}}\,dx', \qquad (16)$$

where t is the time taken for the oscillator to move between positions 0 and x. Now we can apply this to the Duffing oscillator. The energy equation is, in detail,

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{4}\alpha x^4 = E \quad (17)$$

and let's apply this to the situation shown in Fig.7 where the energy E corresponds to $x = A$, where A is the oscillation amplitude.

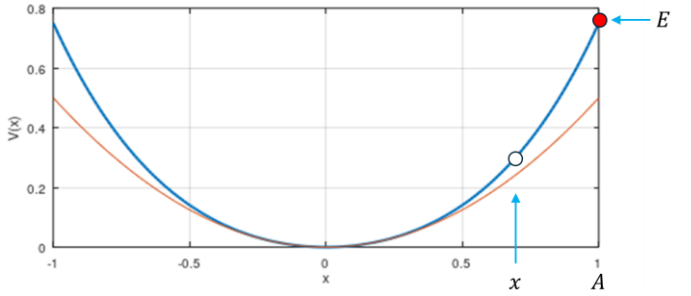


Figure 7.

We therefore have

$$E = \frac{1}{2}kA^2 + \frac{1}{4}\alpha A^4,$$

and the period becomes

$$T = 4\int_0^A \sqrt{\frac{m/2}{(\frac{1}{2}kA^2 + \frac{1}{4}\alpha A^4 - \frac{1}{2}kx^2 - \frac{1}{4}\alpha x^4)}}\, dx'. \quad (18)$$

We shall not even think about hunting for an analytical solution but rather employ numerical integration of eq.18. The results are given in Table.?? where we show excellent agreement between this approximation to T and the actual ODE solution.

2.2.5 Duffing Equation, He's Energy Balance Approach

He's approach [cite] considers both the total energy of the system, and the fact that we are looking for an oscillatory solution. It's advantage over the above energy method is that it gives us an *analytical* expression for the approximate frequency, and it is suggested that this may be valid for large amplitudes.

We start with the expression for the total energy of our system

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + \tfrac{1}{4}\alpha x^4 = E \qquad (19)$$

which is an invariant. We assume the initial conditions $x(0) = A$, $\dot{x}(0) = 0$. Eq.19 then becomes,

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + \tfrac{1}{4}\alpha x^4 = \tfrac{1}{2}kA^2 + \tfrac{1}{4}\alpha A^4$$

i.e.,

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + \tfrac{1}{4}\alpha x^4 - \tfrac{1}{2}kA^2 - \tfrac{1}{4}\alpha A^4 = 0 \qquad (20)$$

which we assume is valid for all times. We then use the following 'trial' function $x(t) = A\cos\omega t$ and substituting this into eq.20 we expect to find a time-dependent error or *residual* $R(t)$ function

$$R(t) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + \tfrac{1}{4}\alpha x^4 - \tfrac{1}{2}kA^2 - \tfrac{1}{4}\alpha A^4 \qquad (21)$$

Performing the substitution we find

$$R(t) = m\omega^2 \sin^2\omega t + k\cos^2\omega t + \tfrac{1}{2}\alpha A^2\cos^4\omega t - k - \tfrac{1}{2}\alpha A^4 \qquad (22)$$

The residual can never be zero for all *t* since eq.2 is nonlinear and our trial function can never satisfy it at every point in time. This method proceeds by *collocation* where we enforce $R(t') = 0$ at one specific time t'. He suggests we set $\omega t' = \pi/4$ which corresponds to 1/8th of the period. Eq.22 then reduces to

$$m\omega^2 + k + \tfrac{1}{4}\alpha A^2 - 2k - \alpha A^2 = 0 \qquad (23)$$

and so we have the frequency

$$\omega = \sqrt{\frac{k}{m} + \frac{3}{4}\frac{\alpha}{m}A^2} \qquad (24)$$

which compares favourably with eq.10. You may be concerned about the above analysis, the choice of co-location point does seem somewhat arbitrary, and it seems that we are not provided with any guidance on why or how it is chosen[1].

[1] Our own research (unpublished) suggests that the co-location point may be obtained by an optimization of eq.22 looking for the minimum of some function of the residual.

2.2.6 Roundup and Results

We have applied several methods to obtain approximations to the frequency of oscillations to the Duffing Equation. Let's compare the results of these, shown to a higher precision than we usually report in Table ??. Then let's reflect on the strengths and limitations of these approaches.

A	T_{soln}	T_{energy} (eq.15)	T_{stiff} (eq.11)	$T_{HeEnergy}$ (eq.18)
0.1	6.259762	6.259762	6.255876	6.259755
0.5	5.768846	5.768846	5.691445	5.765846
1.0	4.768022	4.768022	4.588590	4.749642
10.0	0.736286	0.736290	0.667895	0.720731
100.0	0.074150	0.074158	0.067166	0.072547

Table ??

The first thing to notice is that the approximations are very good over an enormous range of amplitudes. The largest error is for the T_{stiff} calculation just over 9%; it is interesting to note the T_{energy} calculation has an error of around 0.01% at this amplitude. Indeed the energy approximation is clearly the best.

Unfortunately the energy approach requires the numerical integration of a function, eq.15 does not give us an analytical result, i.e., a 'formula'. The stiffness approach gives us eq.10 and eq.12, both tell us that increasing *A* will increase the oscillation frequency and therefore reduce its period. So at once, we have gained some good understanding of the effects of the nonlinearity, even if the numerical predictions have a little error.

We do have some reservations about the stiffness and He's energy approaches (which may have led to their larger errors). The stiffness approach works for symmetric force functions (cubic, quintic, etc.,) but application to non-symmetric force functions (quadratic, quartic) may need some careful thought. The energy balance method looks straightforward, but the choice of collocation point seems arbitrary.

2.3 The Helmholtz Oscillator

2.3.1 Origins and Features

In comparison with the pure cubic nonlinearity characterizing the Duffing equation, here we consider the equation with a pure quadratic nonlinearity, the Helmholtz or 'ear' equation

$$m\ddot{x} = -kx - \beta x^2 \qquad (25)$$

So where did this equation originate? It was invented by the physicist Helmholtz in the 19th Century while looking at the detailed structure of the ear-drum. Like any drum the ear-drum has a thin membrane stretched over a supporting frame or ring, but unlike a modern drum, the membrane is not stretched flat, but is slightly curved, see Fig.8.

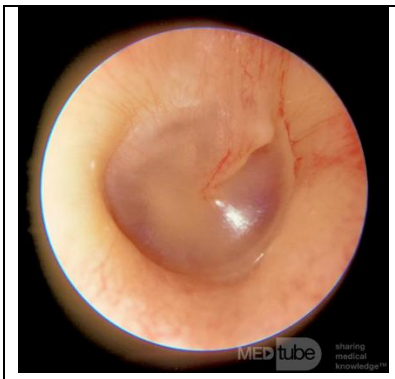


Figure 8.

This shape gives the ear-drum some interesting properties which we shall nor try to understand. Compare the cross-sections from two suggested drumskins in Fig.9. In (a) we have a flat skin and

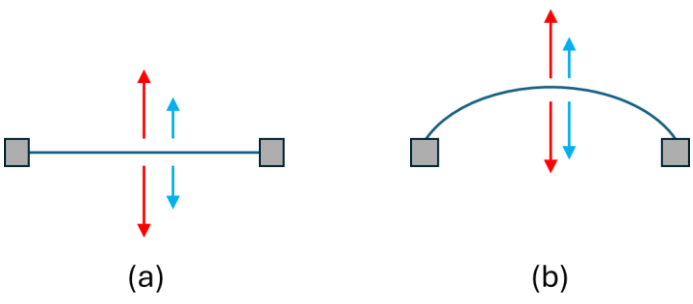


Figure 9.

a force applied up or down (red arrows) would give the same deflexions up or down (blue arrows). In (b) we have a curved skin, and in this case a force down would give a larger deflexion than

the same force up. You might like to do a thought experiment: Cut a tennis ball in half and push it up from the inside; there will be a tiny displacement. But if you push it down from the top, then the displacement will be large since the ball will deform. Actual experiments with tennis balls confirm this.

The question is what sort of non-linearity can explain this, and the answer is a quadratic nonlinearity. Let's see why. The plot in Fig.10 shows the force

$$F(x) = -kx - \beta x^2 \qquad (26)$$

around the origin. If you take a small positive displacement (upwards) then there is a larger increase in force than if you take the same negative (downward) displacement. That's what we want.

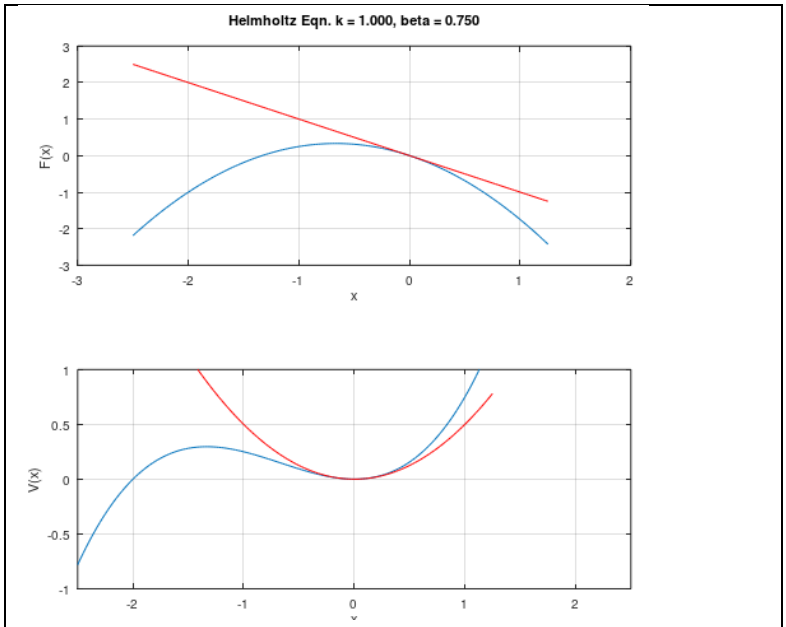


Figure 10.

Figure 10 also shows the potential

$$V(x) = \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3 \qquad (27)$$

which is interesting. There is a maximum at $x = -k/\beta$, so initial values of x less than this will escape the potential well and there will be no oscillation. This maximum in turn defines a finite range of initial x where oscillations may be sustained, and this is confirmed by the phase plot in Fig.11.

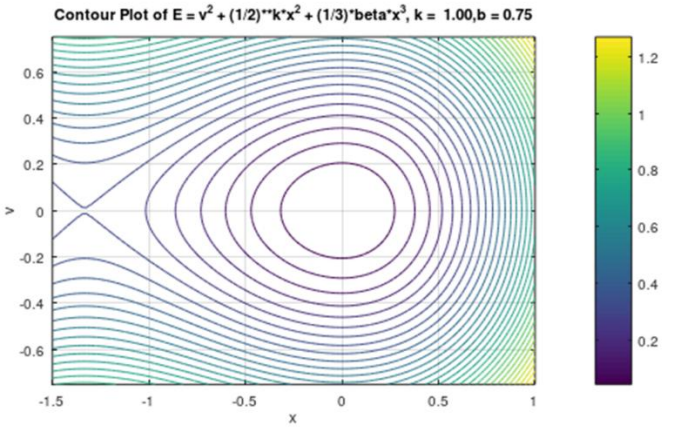


Figure 11.

From the above potential and phase diagram plots, we see that the system is not symmetrical in x unlike the Duffing oscillator, and this asymmetry becomes more pronounced as the amplitude of oscillations increases. We expect the oscillation time-traces to reflect this asymmetry, they will become 'distorted' and will no longer be harmonic; we shall not observe sine curves. Figure.12 shows this where time traces for two initial conditions, $x_{init} = 0.25$ and $x_{init} = 0.60$ are shown. Look at the phase diagram, find these initial conditions (zero velocity) and follow the trajectories.

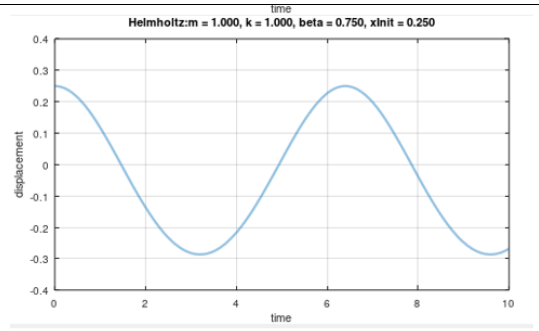


Figure 12.

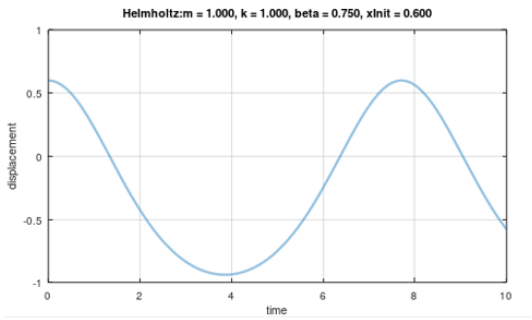


Figure 12 (ctd)

These traces are fundamentally different. The top trace for $x_{init} = 0.25$ appears (almost) harmonic, but on closer inspection its maximum and minimum values are not equal and opposite as they should be; the curve is not symmetric about $x = 0$. This means that we cannot sensibly define the *amplitude* of the oscillation. This is even more pronounced in the trace for $x_{init} = 0.60$ where the magnitudes of peaks and troughs are wildly different. Also you can see that the time-trace shape is far from sinusoidal, the troughs are much flatter than the peaks. This is because the oscillator is close to the 'hump' of the potential. A larger value of x_{init} would take it over the hump; there would be no oscillations, but the time-trace would diverge into nothingness. The takeaway from this is simple, the concept of *amplitude* has no mathematical significance, even though experimenters may report amplitude in their investigations.

2.3.2 Oscillation Frequency

Here we apply the analysis from 2.2.1 to approximate the oscillation frequency. Remember we need to average the value of $\partial F/\partial x$ over the interval experienced by the oscillator. For the Duffing oscillator, the amplitude was a natural (and correct) measure of this interval. In our current case we cannot simply repeat the previous analysis, since amplitude has no meaning.

Instead we much choose another way of characterizing the oscillator interval, and we select the *actual interval* experienced between the smaller value x_- and the larger value x_+. How do we define and obtain these values? Well the potential energy curve provides the answer. If we give an initial location x_{init} and assume zero initial speed, then the oscillator will start off with a potential energy E_{init} corresponding to this location. Since the system is conservative, then the energy will always equal this. So if we start off at rest with $x_{init} = x_+$, then the oscillator will move and come

to rest at x_- where the potential energy is just E_{init}. This is sketched in Fig.13.

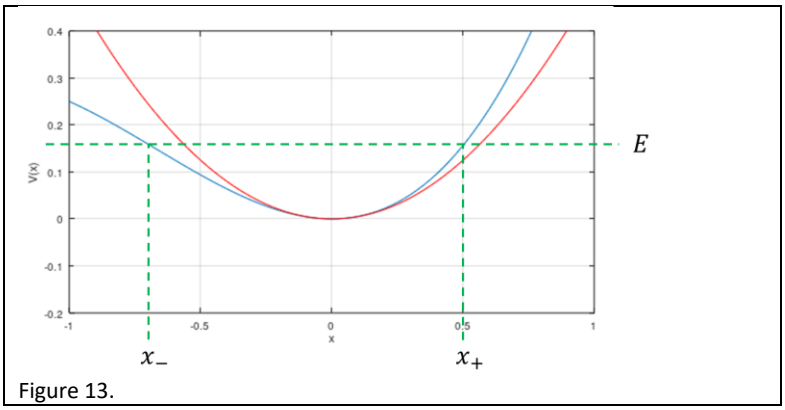


Figure 13.

In order to obtain an expression for the approximate oscillation frequency, we must first determine the interval $[x_-, x_+]$ for a chosen initial x_+. Equating potential energies at the interval ends we have

$$E = \tfrac{1}{2}kx_-^2 + \tfrac{1}{3}\beta x_-^3 = \tfrac{1}{2}kx_+^2 + \tfrac{1}{3}\beta x_+^3 \qquad (28)$$

Since x_+(and system coefficients) are known, this is essentially a cubic equation in x_- which we can solve. Therefore we know the required interval.

Using this interval, we can now calculate the average of $\partial F/\partial x$ which is straightforward since

$$\frac{\partial F}{\partial x} = -k - 2\beta x \qquad (29)$$

is linear in *x*. The expression for the approximate oscillation frequency can then be written

$$\omega = \sqrt{\frac{1}{m}\left(k + \beta(x_+ + x_-)\right)} \qquad (30)$$

You must be careful in interpreting this equation. It suggests that the frequency increases as $(x_+ + x_-)$, which is, of course, correct. But in reality $(x_+ + x_-)$ never increases! This is a consequence of the asymmetry of the energy curve; if you increase x_+ then the absolute value of x_- increases by a larger amount. So the sum $(x_+ + x_-)$ actually decreases. We therefore are led to conclude that

if you increase x_{init}, then the frequency will decrease, and the period will increase.

2.3.3 The Energy equation.

Here we proceed as in Section 2.2.4. Since the force is given by $F = -kx - \beta x^2$ the associated potential is $V(x) = \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3$ so *we are tempted* to write the energy equation as

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3 = E = \frac{1}{2}kA^2 + \frac{1}{3}\beta A^3, \quad (31)$$

and hence calculate the period

$$T = 4\int_0^A \sqrt{\frac{m/2}{(\frac{1}{2}kA^2 + \frac{1}{3}\beta A^3 - \frac{1}{2}kx^2 - \frac{1}{3}\beta x^3)}}\, dx'. \quad (32)$$

This is incorrect! As we have seen, the amplitude A has no theoretical meaning, the oscillation interval is $[x_-, x_+]$ so we must use this interval in the integral eq.32. Since this interval corresponds to half a wavelength, the integral becomes

$$T = 2\int_{x_-}^{x_+} \sqrt{\frac{m/2}{(\frac{1}{2}kx_+^2 + \frac{1}{3}\beta x_+^3 - \frac{1}{2}kx^2 - \frac{1}{3}\beta x^3)}}\, dx'. \quad (33)$$

Results of calculating the period from eq.33 together with accurate numerical solutions of the ODE are shown in Table.xx.

2.3.4 Roundup, Comparisons and Discussion

First, we summarize the data for the stiffness and energy approximations, comparing with an accurate numerical solution of the governing ODE.

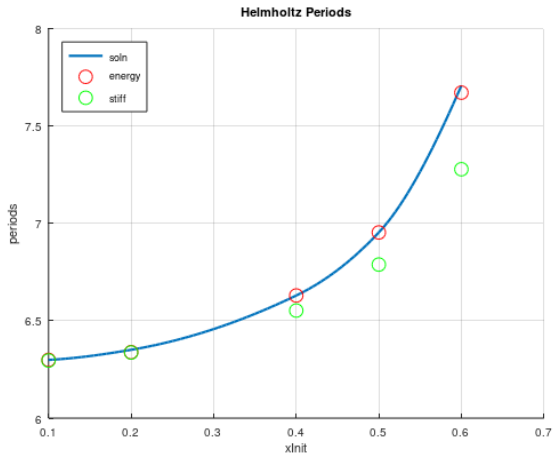
Table?? is organized from left to right according to our computational protocol. For the energy approximation, (i) we choose $x_{init} \equiv x_+$ then (ii) use eq.28 to calculate x_- and finally (iii) use x_+ and x_- to calculate the period from eq.33. The numerical solutions have initial condition x_{init}. For the stiffness approximation, (i) select x_{init} and compute the associated energy *E*. (ii) Use eq.?? to find x_-. This establishes the oscillator's interval of *x* so we can calculate the average $\partial F/\partial x$ and (iii) use eq.?? to calculate the oscillation frequency and then period.

Table?? shows periods calculated from both the stiffness approximation, eq.30 and the energy approximation, eq.33, and

Fig?? shows these as a function of x_{init}. Agreement is good, especially for the energy approximation which has a maximum error of around 0.5%. As expected, the stiffness approximation loses accuracy for larger values of x_{init} as this involves a truncated series.

x_{init}	x_-	T_{soln}	T_{energy}	T_{stiff}
0.10	-0.1053	6.298796	6.298796	6.295711
0.20	-0.2225	6.350691	6.338294	6.336880
0.40	-0.5072	6.629357	6.629355	6.552098
0.50	-0.6910	6.952919	6.952919	6.788172
0.60	-0.9394	7.706476	7.670511	7.277306

Table??



Figure??

While the energy approximation is the best, since it is the most accurate, we suggest that the stiffness approximation is equally good, but for a different reason. The fact is that eq.30 presents us with an analytical expression which guides us in understanding how the frequency changes with the initial conditions. Of course this requires us to have an understanding of the meaning of $(x_+ + x_-)$ which can be gleaned from a glance at the V(x) curve. So there seems a place for both approximations which should be employed before the solution of the ODE is attempted.