# Contents

# Chapter 1
# Introduction to Oscillations

## 1.1 A Brief Introduction

This chapter lays down the basic physics concepts and mathematical language and tools needed to understand the phenomenon of oscillation. Subsequent chapters will apply this material in several interesting scenarios, but we need to start with the basics. We use the term 'oscillation' to refer to a device or system which has a 'periodic motion in time', by which we mean that if we make an observation at a certain time $t$ then at a later time $t + T$ and then at an even later time $t + 2T$ then the observations will be identical. Think of a rotating machine such as an electric motor or a car engine; their shafts return to their initial place (angle) some 2000 – 3000 times per second. So if we measure some property $y$ of the system over time and find that

$$y(t+T) = y(t) \qquad (1)$$

then the system is oscillating with period $T$ in seconds. The *frequency* of oscillation is just $1/T$ in Hertz.

You may have already studied a mass hanging from a spring, and you know that it oscillates up and down, returning to its starting location at regular intervals $T$ secs. You may also have learned that the position of the mass follows an expression like this

$$y(t) = A\cos\left(\frac{2\pi}{T}t\right) \qquad (2)$$

though some folk use the sin function. But this is not the only way to represent these oscillations. You may have seen in a more enlightened text the use of complex numbers with an expression like this

$$\hat{y}(t) = \hat{C}e^{i(2\pi/T)t} \qquad (3)$$

Don't worry, we shall explain what this means in detail in section ??. The thing is that some authors suggest that when you see eq.2 then you 'see' in your mind an oscillating system, whereas when you see eq.3 you do not, and therefore eq.2 is preferred. This is nonsense since what you 'see' depends on your learned experience of the maths. In this chapter we shall make extensive use of complex notation.

There is also another limitation of many texts which make almost exclusive use of the sin or cos function in their discussion without any rationale for this choice. As we shall see in section 2 when we look at traces of oscillation in the real world, we do not see any simple sin or cosine functions, but the traces have more complex shapes.

And there's a third limitation. Most texts discuss 'linear' oscillations and therefore lead us to believe that most real-world oscillations are linear. This is an illusion, and will be addressed in section??

When we describe oscillations, we usually draw a graph of some property against time, e.g., the height of a car above the ground when it bounces after travelling over a bump. We shall see lots of these graphs below. But there's another way of plotting oscillations shown in Fig.1. Here we plot a pair of variables (such as position and velocity) on a closed curve. Time is represented implicitly as the system travels around the curve. During one cycle of oscillation of period $T$ secs, the oscillator will start at point A, go around the curve as shown by the arrows, and return to point A. It turns out that all oscillators can be described like this. In some cases the curve approximates a circle quite closely, we shall see this in section ??.

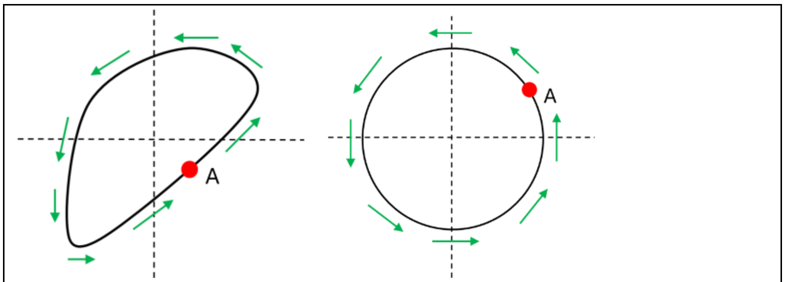
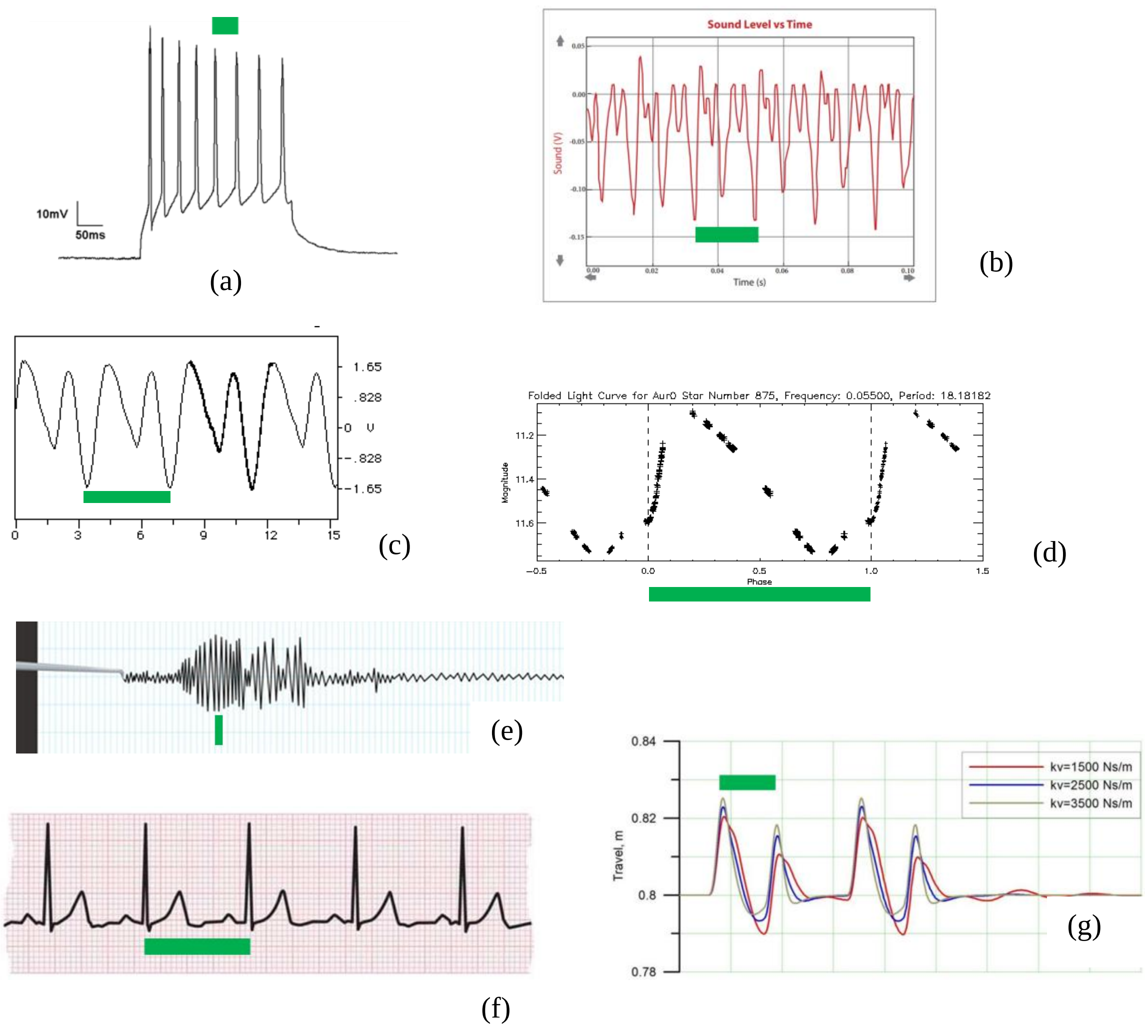



Figure 1. Plots of e.g., position and velocity as time increases. Green arrows show the direction of time.

## 1.2 Oscillations in the Natural World

Figure 2 presents a selection of oscillations we may observe in the natural world; note that most of these do not have a sine waveform.



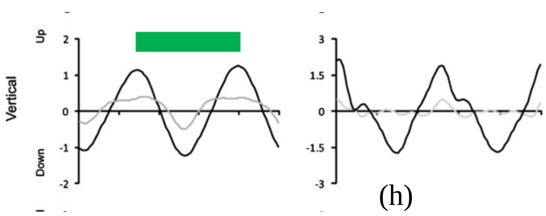




Figure 2. (a) Spikes from a single nerve cell, (b) sound from a guitar string, (c) Sound from a bassoon, (d) light from a Cepheid variable star, (e) earthquake recording, (f) electrocardiogram, (g) car suspension, (h) height of a walking human,

## 1.3 What you Already Know

Here we shall present some short notes on material you will have already encountered. While the descriptions are , they are certainly not *complete*, (more about this later).

### 1.3.1 Mass on a Spring

This is the archetypal ‘simple harmonic oscillator’ (SHO) shown in Fig.3. On the left is an unstretched spring of length $L_0$ and spring constant (‘stiffness’) $k$.

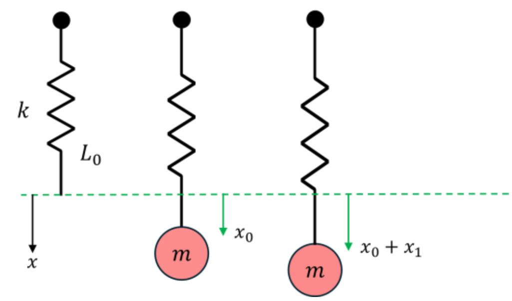



Figure 3. Mass on a spring. Left, unstretched spring; centre, mass added and spring stretches to equilibrium, right, mass pulled down.

Hanging a mass on the end makes the spring stretch down to $x_0$ and here the force of gravity is balanced by the upward force from the spring

$$mg - k(L - x_0) = 0. \qquad (4)$$

If we then displace the mass down by an additional amount $x_1$ then the total force in the positive-$x$ direction is

$$mg - k(L - x_0) - kx_1 = -kx_1. \qquad (5)$$

Using Newton’s second law, we find the acceleration of the mass is just

$$\frac{d^2x_1}{dt^2} = -\frac{k}{m}x_1. \qquad (6)$$

This describes oscillations of the mass *around its equilibrium position* $x_0$. To solve this, we need a function whose second derivative is -1 times that function. A cosine will do nicely, so we try $x(t) = A\cos\omega t$; substitution into eq.6 gives expressions for the angular frequency and period of oscillation

$$\omega = \sqrt{\frac{k}{m}}, \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad .(7)$$

Here $\omega = 2\pi/T$. Eqs7 do not contain the amplitude $A$ of oscillation, so its period is independent of amplitude which is a special property for this type of oscillator. Fig.4 shows the mass located at time $t = 2$ secs for a system with period $t = 2$ secs. You will see that the amplitude is -1.

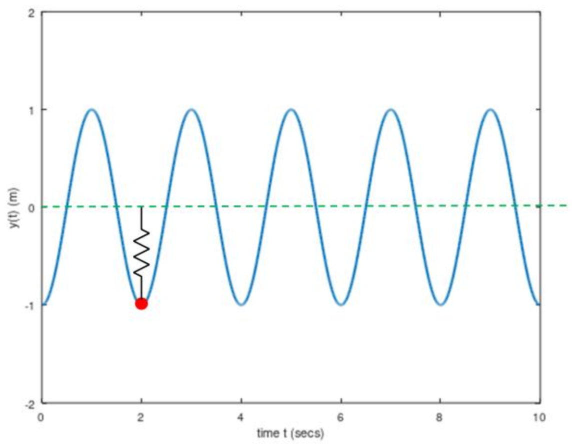



Figure 4. Displacement time-trace for oscillating mass.

Instead of starting with a Newton's 2nd and setting up an ordinary differential equation (ODE) we can analyse systems in terms of energy. For a mass-spring system we have for the kinetic energy (KE) stored in the mass and for the potential energy (PE) stored in the spring

$$KE = \tfrac{1}{2}mv^2, \qquad PE = \int_0^x kx\,dx = \tfrac{1}{2}kx^2. \qquad (8)$$

Substituting $x(t) = A\cos\omega t$ and $v(t) = -\omega A\sin\omega t$ we find that

$$KE = \tfrac{1}{2}m(-\omega A\sin\omega t)^2$$

$$= \tfrac{1}{2}kA^2\sin^2\omega t, \qquad (9)$$

and

$$PE = \tfrac{1}{2}kA^2\cos^2\omega t. \qquad (10)$$

These energies change with time, and it easy to see that they oscillate twice as fast as the displacement or velocity, e.g.

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, see Fig.5. Now we can form an expression for the total energy $E$

$$E = KE + PE = \tfrac{1}{2}kA^2(\sin^2\omega t + \cos^2\omega t) = \tfrac{1}{2}kA^2, \quad (11)$$

since $\sin^2\theta + \cos^2\theta = 1$. So we see that energy is conserved for this system. Also, if we denote the average value of a quantity like this $\langle quantity \rangle$ then $\langle KE \rangle = \langle PE \rangle = \frac{1}{4}kA^2$ since $\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = \frac{1}{2}$, so on average the total energy is shared equally between KE and PE, and as shown in Fig.5 energy sloshes between PE and KE.

So far, all this energy talk is rather descriptive, some may even say boring. But things become much more interesting if we take the energy equation and differentiate it,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \qquad (12)$$

$$\frac{dE}{dt} = m\frac{dx}{dt}.\frac{d^2x}{dt^2} + kx\frac{dx}{dt}. \qquad (13)$$

Please glance at this footnote[1] if you're uncertain about this. Now we know that the total energy $E$ is conserved.

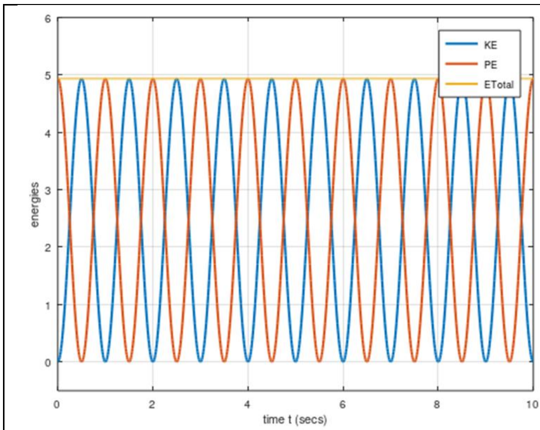



Figure 5. KE and PE variation with time together with total energy.

Since energy is conserved, we must have $\frac{dE}{dt} = 0$, so eq.13 gives us, assuming that $dx/dt \neq 0$

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[1] First $\frac{d(v^2)}{dt} = 2v\frac{dv}{dt}$ by the chain rule, where $v = \frac{dx}{dt}$ of course, and also using the chain rule we have $\frac{d(x^2)}{dt} = 2x\frac{dx}{dt}$

$$m\frac{d^2x}{dt^2} = -kx, \qquad (14)$$

which is of course the ODE eq.6 we obtained from Newton's $2^{nd}$. So the huge takeaway here is that we can derive the ODE either by considering *forces* or *energy*, and there are many cases where using energy is much easier. Let's formalize this; it will reduce work in the future.

### 1.3.2 An aside concerning Energy.

Apologies for this subsection. The whole section was about what you already know, but I can't resist just pushing this knowledge a bit. *Pace lector*. We have just seen in eq.12 and eq.13 an important link between understanding the dynamics of a system (expressed as an ODE) and its energy formulation. Let's try to generalize this approach (to make future work easier). We can write the system energy like this, as a sum of kinetic and potential energies,

$$E = \tfrac{1}{2}\alpha\dot{\psi}^2 + \tfrac{1}{2}\beta\psi^2. \qquad (15)$$

Yikes! We have introduced the strange symbol $\psi$; what does this represent? Well, the idea is that eq.15 could be applied to many scenarios. In one, $\psi$ could be just $x$, the position of a mass on a spring; in another it could be $\theta$, a rotation of a pendulum. Or it could be a temperature, or a magnetic field. The important point is to *recognize* the structure of eq.15, which may be applicable to many physical systems.

So we work out the temporal derivative of $E$ and find

$$\dot{E} = \alpha\dot{\psi}\ddot{\psi} + \beta\psi\dot{\psi}, \qquad (16)$$

which, due to energy conservation, $\dot{E} = 0$, gives us

$$\ddot{\psi} + \frac{\beta}{\alpha}\psi = 0, \qquad (17)$$

assuming that $\dot{\psi} \neq 0$ and therefore leads to an oscillation frequency

$$\omega = \sqrt{\frac{\beta}{\alpha}}. \qquad (18)$$

All we need to do is identify $\alpha, \beta$ for a specific situation and invoke eq.15 and eq.18. This is the basis of the 'energy method'. We'll see this in action in the following sections.

### 1.3.3 Mass on 2 Springs

This is a very simple system, a mass is free to move in a line on a frictionless horizontal surface, see Fig.6.

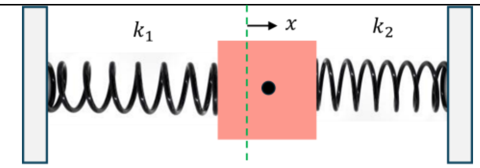



Figure 6. Oscillating mass with springs of different stiffness.

The mass is displaced $x$ to the right, the left spring pulls to the left with force $-k_1 x$ and the right spring pushes to the left with force $-k_2 x$ so we have

$$m\frac{d^2x}{dt^2} = -(k_1 + k_2)x, \quad \omega = \sqrt{\frac{k_1 + k_2}{m}}.$$

### 1.3.4 Floating Bob Wave Energy Convertor

This is a simple system; a bob is floating in water. At equilibrium, its weight is balanced by an upward force equal to the weight of the water displaced (Archimedes principle), see Fig.7. If we displace the bob downwards by distance $y$ then the submerged volume increases by $Ay$ where $A$ is the cross-sectional area of the bob. The mass of the additional displaced water is just $\rho A y$ where $\rho$ is the water density and its weight is just $\rho g A y$ where $g$ is gravitational acceleration, so we now have a net upwards force. We therefore have

$$m\frac{d^2y}{dt^2} = -(\rho g A)y, \qquad \omega = \sqrt{\frac{\rho g A}{m}}. \qquad (19)$$

This can be simplified by using an expression for $m$ but here's not the place.

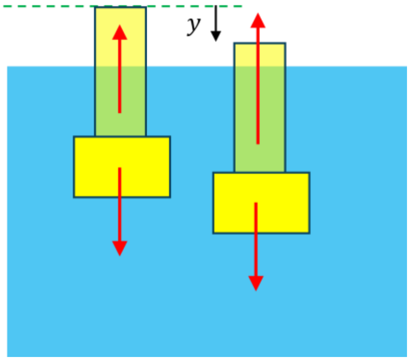


Figure 8. Floating Bob Wave Energy Convertor (WEC).

### 1.3.5 The Pendulum

This is shown in Fig.8. Here we apply Newton's 2nd but in rotation. If the body has moment of inertia $I$ and we apply a torque $\tau$ about a point, then the body experiences an angular acceleration given by

$$I\frac{d^2\theta}{dt^2} = \tau. \qquad (20)$$

In this case downward force on the bob due to gravity produces a torque around the pivot point

$$mgL\sin\theta, \qquad (21)$$

since torque is the product of the force and the lateral distance to the centre of rotation. The moment of inertia of the pendulum about its pivot point is just $mL^2$ so the equation of motion for the bob's rotation becomes

$$mL^2\frac{d^2\theta}{dt^2} = -mgL\sin\theta, \qquad (22)$$

which becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta. \qquad (23)$$

Now *for small* $\theta$ we have $\sin\theta \approx \theta$ so we finally arrive at

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta, \qquad \omega = \sqrt{\frac{g}{L}}. \qquad (24)$$

Alternatively, we could use the energy approach. The KE is just $\frac{1}{2}I\dot{\theta}^2$ and the PE is $mgL(1-\cos\theta) \approx \frac{1}{2}mgL\theta^2$ since $\cos\theta = 1-\frac{1}{2}\theta^2$, so we use the energy formula with

$$\alpha = mL^2, \quad \beta = mgL, \qquad (25)$$

which leads to eq.24.

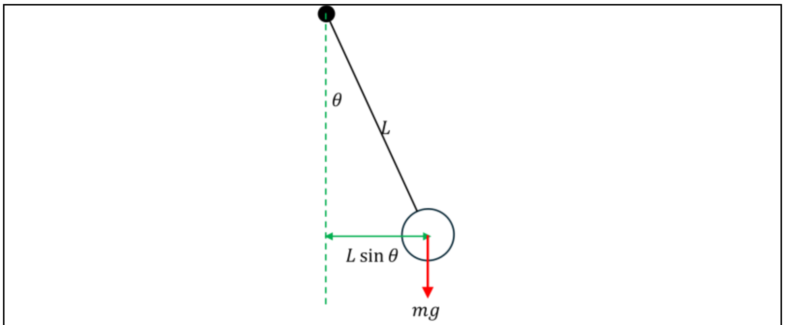



Figure 8. Pendulum Bob.

## 1.3.6 Water in a U-tube

Have a look at Fig.9 which shows water in a U-tube displaced upwards by distance $y$ from the equilibrium position (green dashed line). The tube has cross-sectional area $A$ and the water has density $\rho$ and the total column length $L$. Let's put forward two arguments, the first using Newton's $2^{nd}$ and the second using the energy approach.

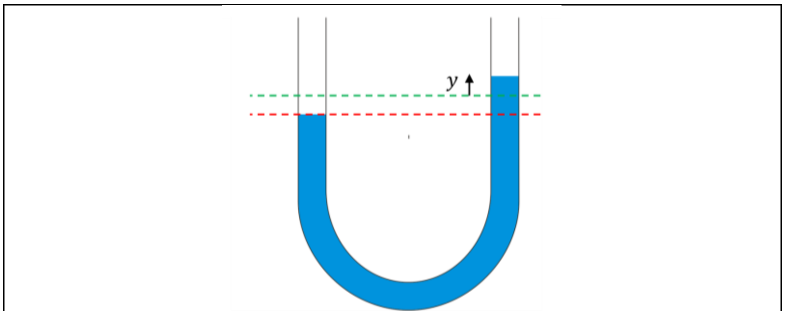



Figure 9. Water in a U-Tube.

There is a force due to the additional length of water on the right which is just $-(A\rho g)2y$ , use the red dashed line to check this, and this acts to accelerate a mass. But which mass, you may think since the force is exerted on the water below the additional length, then we should use that mass. But this is incorrect. Ask yourself what mass is actually accelerating due to this force? Clearly the mass of the whole water column. We therefore have

$$(\rho AL)\frac{d^2y}{dt^2} = -(2A\rho g)y, \qquad \omega = \sqrt{\frac{2g}{L}}. \qquad (26)$$

Now let's apply the energy approach. The increase in PE of the displaced water from its equilibrium position (green dashed line) comes from taking a volume of water *Ay* from the left arm, and *raising it a height Ay* and putting it on top of the water in the right tube. We therefore have

$$PE = (\rho gAy)y \qquad = \rho gAy^2.$$

The kinetic energy of the moving mass of water is just

$$KE = \tfrac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \tfrac{1}{2}\rho AL\left(\frac{dy}{dt}\right)^2, \qquad (27)$$

so we have

$$\alpha = \rho AL, \qquad \beta = 2\rho gA, \qquad (28)$$

which again leads to eq.26.

### 1.3.7 Diatomic Molecule.

Let's work this situation using the energy method. The situation is shown in Fig.10 The diatomic molecule is modelled as two different masses connected by a single spring. You may think this is a rather unusual system, since it is not 'tied' to any fixed point in space. So it could be rotating or moving through space as well as vibrating, or a combination of any or all of these degrees of motion. So we assume it is not moving, the centre of mass is fixed (i.e., assumed 'tied') and it is not rotating. So we consider only vibrations, around its centre of mass. This is equivalent to assuming there is no 'external' force.

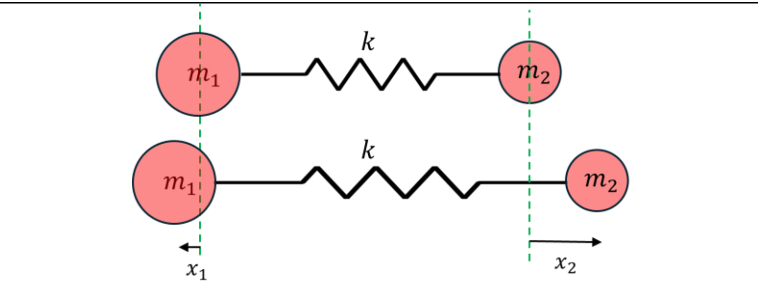



Figure 10. Diatomic Molecule vibration.

Since the centre of mass does not move, we must have $m_1 x_1 + m_2 x_2 = 0$. Now let's get our PE and KE and use this constraint:

$$PE = \tfrac{1}{2} k (x_1 + x_2)^2 = \tfrac{1}{2} k x_1^2 \left(1 + \frac{m_1}{m_2}\right)^2 , \qquad (29)$$

$$KE = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 = \tfrac{1}{2} m_1 \left(1 + \frac{m_1}{m_2}\right) , \qquad (30)$$

so we have

$$\alpha = m_1 \left(1 + \frac{m_1}{m_2}\right), \qquad \beta = k \left(1 + \frac{m_1}{m_2}\right)^2 , \qquad (31)$$

so we find the oscillation frequency is

$$\omega = \sqrt{\frac{\beta}{\alpha}} = \sqrt{k \left(\frac{m_1 + m_2}{m_1 m_2}\right)} = \sqrt{\frac{k}{\mu}} , \qquad (32)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is called the *reduced mass* of the molecule, just replacing two parameters with one. Nature likes simplicity.

**Example 1.1** Let's consider a molecule of Hydrogen Chloride. The masses are $m_H = 1 \times 1.67 \times 10^{-27}$ kg and $m_{Cl} = 35 \times 1.67 \times 10^{-27}$ kg giving a reduced mass of $\mu = 1.62 \times 10^{-27}$ kg which is nearly the same as the hydrogen mass. So we it looks as though the chlorine atom is behaving like a brick wall. The spring constant is 516 N/m so we calculate

$$\omega = 5.65 \times 10^{14} s, \qquad f = 9.00 \times 10^{13} Hz$$

which is in the infra-red region of the spectrum. So if we shine electromagnetic waves into a tank of HCl then we shall see an absorption band in the infra-red region at this frequency.

## 1.4 Some Underlying Mathematics

We mentioned earlier that while solutions presented in section 1.3 were correct, they were not necessarily complete. Here we address this limitation.

### 1.4.1 Various ways to express harmonic motion

We noted that to solve eq.6 either a sine or a cosine function would do, and we selected a cosine (in order to make the equations less cluttered). We could have selected a sine, but we should have selected both! The idea is that if a cosine is a solution and a sine is a solution, then added together they would also be a solution like this

$$y(t) = A\cos\omega t + B\sin\omega t \qquad (33)$$

where the coefficients $A$ and $B$ allow us to take different mixes of the two functions. Go ahead and check this *is* a solution by substitution into eq.6. Now we can use trigonometric identities to write eq.33 in the alternative form

$$y(t) = D\cos(\omega t + \varphi) \qquad (34)$$

where the variable $\varphi$ is the 'phase' of the oscillation. Of course another alternative solution is

$$y(t) = D\sin(\omega t + \varphi') \qquad (35)$$

where we need a different phase. Let's try to get our heads around the meaning of phase and plot out eq.34 for various values of phase $\varphi$. Note that $\varphi$ is just a specified angle (in radians), since at $t = 0$ the above expression will become $y(0) = D\cos(\delta)$. If you want to visualize a phase in degrees (°), just calculate $\varphi \times (180/2\pi)$, but never use degrees in an expression!

We plot eq.34 for a number of different phases in Fig.11, each plot has a red reference function with phase set to zero. In Fig.11(a) we have $\varphi = \pi/3$ (30°) and we see the plot is shifted *to the left*. This is easy to understand, $\omega t$ must now be smaller for the argument of the cos to have the same value, hence $t$ is smaller, a shift to the left. In Fig.11(b) we have $\varphi = -\pi/3$ (-30°) and the function is shifted to the right. Finally in Fig.11(c) we have $\varphi = \pi/2$ (90°) which turns out to be an important value for phase. Here we have a left shift, and we say the green curve *lags* the red curve by $\pi/2$ .

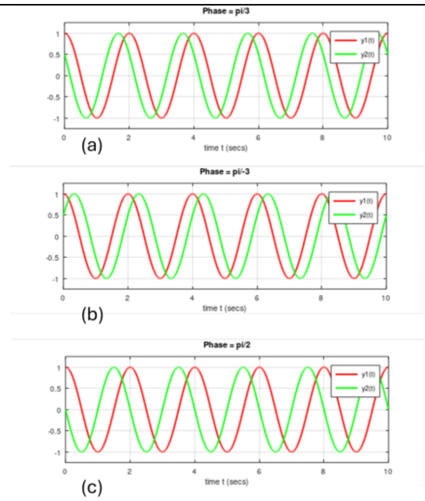



Figure 11. Cosine wave with three phases: (a) 30°, (b) -30°, (c) 90°.

So you should have a fairly good understanding of what the phase angle is, but we have not really addressed the question of why we have introduced it at all! Well note that all the 'various ways' of expressing SHM mentioned above have *two arbitrary* coefficients. Why? Well, it's because we are dealing with a $2^{nd}$-order ODE, and mathematically, there must be an arbitrary coefficient for each order. We shall see how these coefficients are specified by the *initial conditions* (the state of the system at $t = 0$, in section 1.4.4.

### 1.4.2 Linear and Non-linear Systems, Superposition

We have mentioned that linear systems are straightforward to solve, but non-linear systems are not; also that most systems in the natural world are non-linear. So what's the difference? Consider two springs, for a *linear* spring, the restoring force is proportional to displacement

$$F(y) = -ky. \qquad (36)$$

Now this is not true for a non-linear spring, e.g. the force expression may have an additional cubic term

$$F(y) = -ky - \alpha y^3. \qquad (37)$$

We know that eq.36 leads to eq.6 which has a solution of the form $y(t) = A\cos\omega t$, but what happens when we set up the ODE for the non-linear force and try a solution of this form. We have

$$\ddot{y}(t) = -\frac{k}{m}y(t) - \frac{\alpha}{m}y^3(t), \qquad (38)$$

$$-\omega^2 A\cos\omega t = -\frac{k}{m}A\cos\omega t - \frac{\alpha}{m}A^3\cos^3\omega t, \qquad (39)$$

and following some tedious algebra we have

$$-\omega^2\cos\omega t = -\frac{k}{m}\cos\omega t - \frac{\alpha}{4m}A^3\cos\omega t - \frac{\alpha}{4m}A^3\cos 3\omega t. \qquad (40)$$

Now we're in trouble, note the appearance of the term in $\cos 3\omega t$ on the right. So we are unable to 'cancel out' the cos terms, i.e. find a solution which works for all values of $t$. So we conclude that this non-linear system does not undergo SHM. We'll return to this in section 1.6.4.

So to linear systems. In section 1.4.1 we mentioned that both a cos and a sin function solved the ODE and hinted that a sum should satisfy the ODE. Let's make this more concrete. Let's take eq.6 and say that we have found two solutions $y_1(t)$ and $y_2(t)$ so we have

$$m\frac{d^2y_1}{dt^2} = -ky_1,$$

$$m\frac{d^2y_2}{dt^2} = -ky_2$$

Adding these together we find, using the fact that a sum of derivatives is the derivative of the sum,

$$m\frac{d^2(y_1 + y_2)}{dt^2} = -k(y_1 + y_2), \qquad (41)$$

which is an ODE for the sum $y_1 + y_2$ which clearly has a solution of the form we have been using. This works because the right-hand sides of the above equation are linear in $y$ corresponding to a linear spring.

### 1.4.3 An Alternative view of the ODEs

This material is perhaps a little advanced and could initially be skipped. We have been using an ODE of the form $d^2y/dt^2 = f(y)$ to describe our oscillators. This is a *second-order* ODE since the second derivative of $y$ appears in the expression. It's interesting to look at an alternative way of expressing this 2nd-order ODE and indeed we can generalize this to any *n*th-order linear ODE.

We start with the 'fundamental theorem of algebra' which states that any *n*th-order polynomial

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (42)$$

can be factored into

$$a_n (z - r_1)(z - r_2) \dots (z - r_n). \qquad (43)$$

So if we take the *n*th-order ODE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 = 0, \quad (44)$$

then since differentiation by $t$ commutes with multiplication, we can use the equality of eq.42 and eq.43 to rewrite this ODE as

$$a_n \left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right) \dots \left(\frac{d}{dt} - r_n\right) x = 0. \qquad (45)$$

Now the last term invites an immediate solution

$$\left(\frac{d}{dt} - r_n\right) x = 0, \qquad \Rightarrow \frac{dx}{dt} = r_n x, \quad (46)$$

and since any of the terms in eq.45 can be 'rotated' to the right, we see that the *n*th-order ODE can be re-written as *n* 1st-order ODEs. This can be quite useful, and we shall soon see how. The takeaway from this, is that the dynamics contained in our 2nd-order ODE is equally contained in the dynamics of two 1st-order ODEs.

### 1.4.4 Initial Conditions and Superposition

We have seen how a function of the form $y(t) = A\cos(\omega t + \varphi)$ satisfies the ODEs for a range of physical systems, and we know that the value of the angular frequency $\omega$ is specified by the physics of the system; for a mass-spring system it is specified by $m$ and $k$. But we have been silent about the amplitude $A$ and the phase $\varphi$, how are these fixed for an actual system we may be investigating in the lab? Well, they are given by the *initial conditions* (ICs) of the system; how it is configured at $t$=0. We need 2 initial conditions since we have two unknowns $A$ and $\varphi$. How can we impart these ICs, alternatively what must we do to get our oscillator going? Well the first thing is to give it an initial displacement, which will fix the value of $A$ as this displacement. We do this, in the case of the mass-spring system, by dragging the mass down, then releasing it, so at the time of release $t$=0 it takes on the displacement we are giving it. But at the time of release, we have specified another variable, the *velocity* of the mass. This is usually zero. Let's see how this works to fix $A$ and $\varphi$.

Assume that at time $t$=0 we release the mass with displacement $y_0$ and zero velocity. We have

$$y(t) = A\cos(\omega t + \varphi), \quad v_y(t) = -\omega A\sin(\omega t + \varphi), \quad (47)$$

and at $t$ = 0 we have

$$y(0) = A\cos(\varphi) = y_0\,, \quad v_y(0) = -\omega A\sin(\varphi) = 0. \quad (48)$$

The condition of zero velocity at $t$ = 0 means that $\varphi = 0$, and using this in the condition for y(0) gives us $A = y_0$, so our final solution with ICs incorporated becomes

$$y(t) = y_0\,\cos(\omega t). \qquad (49)$$

### 1.4.5 Oscillations near bottom of Potential Well

We saw above that when the potential energy has a quadratic form, then SHO results. We have for $PE = U(x)$

$$U(x) = \tfrac{1}{2}kx^2, \;\; F = -\frac{dU}{dx} = -kx, \qquad (50)$$

which is the required restoring force proportional to $x$. Now if we have a potential which is not quadratic but has a minimum, we can approximate the potential near the minimum with a quadratic, glance at Fig.12. The blue curve is the non-quadratic with a minimum at $x_0$ and the red curve is a quadratic chosen to match the potential at the minimum point.

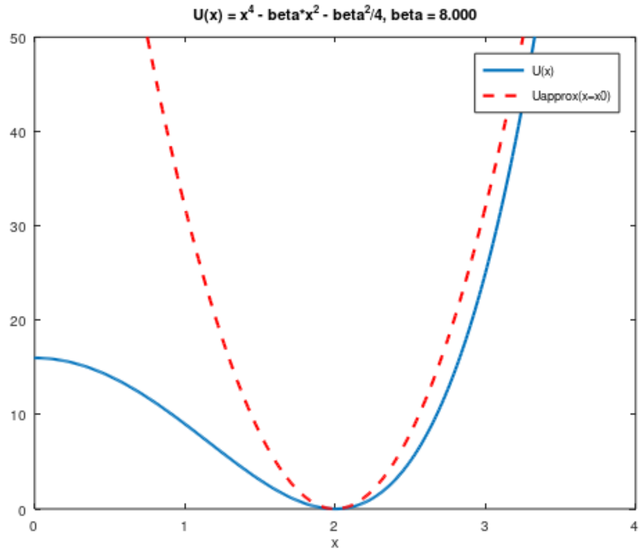


Figure12. Non-quadratic potential (blue) with quadratic approximation (red dash) located at the minimum of blue.

Let's make a Taylor expansion of our non-quadratic potential $U(x)$ about $x_0$

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x=x_0} (x - x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0} (x - x_0)^2 + \cdots. \quad (51)$$

From Fig.12 we have $U(x_0) = 0$ and $dU/dx = 0$ since we are at the minimum which leaves

$$U(x) = \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0} (x - x_0)^2. \quad (52)$$

This is our locally quadratic potential, with effective spring constant

$$k = \left.\frac{d^2U}{dx^2}\right|_{x=x_0}, \quad (53)$$

so in general we have

$$\omega = \sqrt{\frac{1}{m}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}} \quad \text{or} \quad \omega = \sqrt{-\frac{1}{m}\left.\frac{dF}{dx}\right|_{x=x_0}}. \quad (xx)$$

**Example 2.** Let's take an example of the potential

$$U(x) = x^4 - \beta x^2 + \beta^2/4$$

which is shown in Fig.?? This has a minimum at $x_0 = \sqrt{\beta/2}$. Simple calculus shows that $\ddot{U}(x - x_0) = 4\beta$, so our approximating potential is just

$$U_{approx}(x) = \ddot{U}(x - x_0)(x - x_0)^2 = 4\beta(x - x_0)^2$$

## 1.4.6 Extending the ODEs: Damping

Most oscillators in the real world do not continue indefinitely with a constant amplitude as the examples presented above suggest. They will often encounter fluid resistance which removes energy from the system (dissipated ultimately as heat) and causes a steady decay in the amplitude of oscillation. Typical traces for such a *damped* oscillator are shown in Fig.13

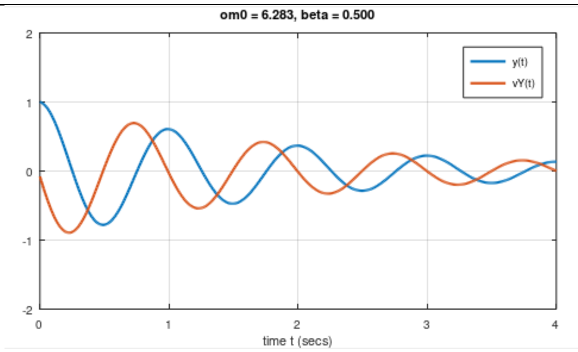



Figure 13. Time-trace (blue) and velocity trace (red) for a damped harmonic oscillator.

Clearly the amplitudes of both displacement and velocity reduce with time. This will be discussed in detail in section 1.8; here we only wish to extend the underlying ODEs to take damping into account.

We must consider the source of damping; this is an additional frictional force due to the object's motion through the surrounding fluid. How this force actually works is shown in Fig.14 where we see a moving object at two times; at each time the velocity of the object is shown by the green arrow; it has reversed direction.

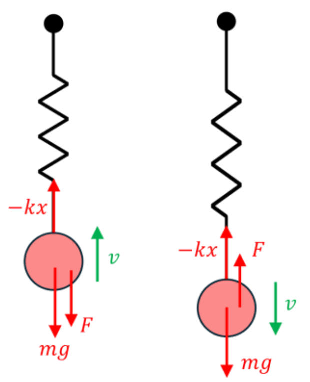


Figure 14. Forces on an oscillating mass with damping. Green arrows show the velocity at two instants, the red arrow *F* shows the friction force in the opposite direction to the velocity.

In this figure we have supplemented the existing forces *mg* due to gravity and -*kx* due to the spring with a frictional force *F*. The direction of this force has to be opposite to the velocity *v* of the object, but how does it depend on the velocity? This needs to be established by experiment, and there appear a couple of alternatives. For air, and at small speeds in water, friction is proportional to velocity, so we write

$$F = -bv, \qquad (55)$$

where the coefficient *b* depends on the size of the object and properties of the fluid (e.g. viscosity). For larger speeds in water friction is proportional to velocity squared, so we have

$$F = -bv^2. \qquad (56)$$

Which do we choose? We nearly always choose eq.55 since eq.56 is non-linear which makes a solution beyond our grasp (note the squared ‘thing’ in eq.56 is not a constant, but a system variable) and this would result in a non-linear differential equation.

So we supplement eq.6 with eq.55 and write down the ODE for our damped system,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{b}{m}\frac{dx}{dt}, \qquad (57)$$

or in the alternative simpler form

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - 2\beta\frac{dx}{dt}, \qquad (58)$$

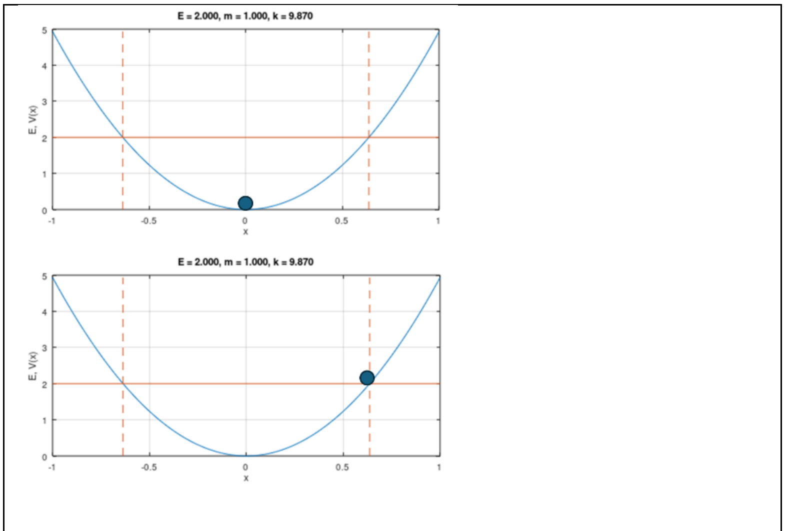
where $\omega_0^2 = k/m$ and $\beta = b/2m$.

# 1.5 Energy

## 1.5.1 Basic Ideas

When a system oscillates it will cross back and forwards around its *equilibrium* position. If there is no damping, it will do this endlessly, but with some damping it will converge to and come to rest at its equilibrium position. Take a pendulum with damping; when displaced it will cross back and forwards around its equilibrium position (the vertical) and will eventually come to rest in this vertical position. So that gives us a hint of what the 'equilibrium position' really means.

Now we know that a system can be described by the sum of potential and kinetic energies, which is constant if energy is conserved. At the point of minimum potential energy, the system will have its greatest kinetic energy. Let's consider a ball on a track which we can think of as its PE curve[2] see Fig.15.




[2] Sure this is an approximation, but it's not a bad one, and many textbooks use this example.

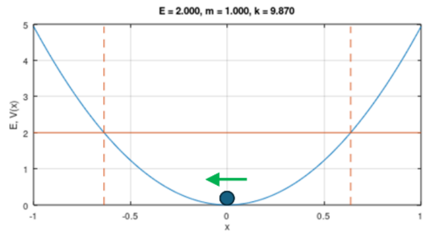



Figure 15. Ball on a track which models its potential energy. Top shows ball at rest, centre a displacement giving it PE, bottom when released the PE is totally converted to PE.

So if we start off with the ball at rest, located at the PE minimum, and give it a small displacement up the curve, we have done work, and given it some PE. If you let it go it will then move down the potential hill towards the PE minimum (where it has maximum speed and therefore KE) and would continue up the other side of the PE curve. This would repeat and the ball would oscillate. If you added damping, then the ball would lose velocity over time, and it would eventually come to rest at the minimum of the PE and 'be at equilibrium'. So we see the equilibrium is *located at the minimum* of the PE curve. See Fig.16.

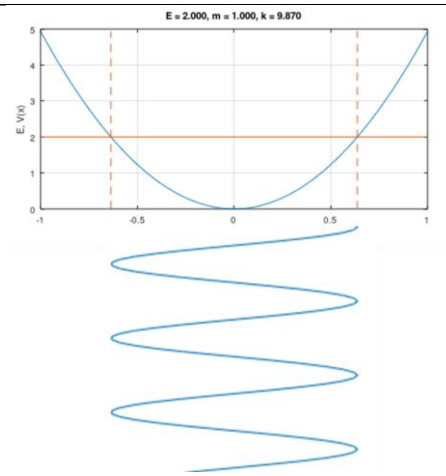



Figure 16. Potential energy curve and time-trace of the mass displacement as it repeatedly crosses the equilibrium (zero) position.

Simple systems, like those discussed in section 1.3 will have a single minimum and quite often the form of the PE curve is *quadratic* i.e., $V(x) = \frac{1}{2}kx^2$ where $k$ is a constant. Other curves may have a different functional form, we have seen how to deal with this in section 1.4.5. Other systems may have multiple minima in their PE curves, and we expect that they could display different oscillation regimes, one for each minima. But that's getting too advanced too quickly.

### 1.5.2 Energy Equation

Let's think of a particle of mass *m* moving in 1D and has a position $x(t)$ which is varying in time, and it is subject to a force $x(t)$ which varies with position. The ODE describing its motion is just $m\ddot{x} = F(x)$. The *kinetic energy* of the particle is

$$T = \frac{1}{2}m\dot{x}^2, \qquad (59)$$

and its *potential energy* is

$$V(x) = -\int_a^x F(x')dx', \qquad (60)$$

where $a$ is an arbitrary constant; we often choose $a = 0$. Now let's calculate the rate of change of KE,

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \dot{x}m\ddot{x} = \dot{x}F(x). \qquad (61)$$

But we know that $F(x) = -dV/dx$ so the above equation becomes

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = -\frac{dx}{dt}\frac{dV}{dx} = -\frac{dV}{dt}. \qquad (62)$$

So the rate of change of KE is equal and opposite to the rate of change of PE, so their sum must be constant, which is often written

$$T + V = \text{const.} \qquad (63)$$

Such a system is called *conservative*. Now let's see what happens if the force *F* depends on something other than position, such as velocity. Think of a mass on a spring with damping; the ODE for this system is

$$m\ddot{x} = -kx - b\dot{x}. \qquad (64)$$

The sum of the KE and PE is still $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ and its rate of change is

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\right) = m\dot{x}\ddot{x} + kx\dot{x}$$

$$= \dot{x}(m\ddot{x} + kx) = -b\dot{x}^2 < 0, \qquad (65)$$

so this is negative, and energy is being lost or *dissipated* by damping. So here $T + V \neq \text{const}$ and we do not have a conservative system.

### 1.5.3 The Energy Integral and calculation of Period

Here we shall obtain an important and useful result which will allow us to find the period of *any* system if we know its energy. Starting with

$$\frac{1}{2}m\dot{x}^2 + V(x) = E, \qquad (66)$$

we find that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2(E-V)}{m}}, \qquad (67)$$

hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2(E-V)}},$$

which we integrate

$$t = \int \frac{dt}{dx}dx = \pm\int\sqrt{\frac{m}{2(E-V)}}\,dx. \qquad (68)$$

We can represent this graphically for a quadratic potential. Fig.17 shows such a potential together with a particular choice of $E$. The limits of $x$ for the oscillation are shown. Also shown is the value of the integrand $\sqrt{m/2(E-V)}$ , and the integral is indicated as the shaded area.

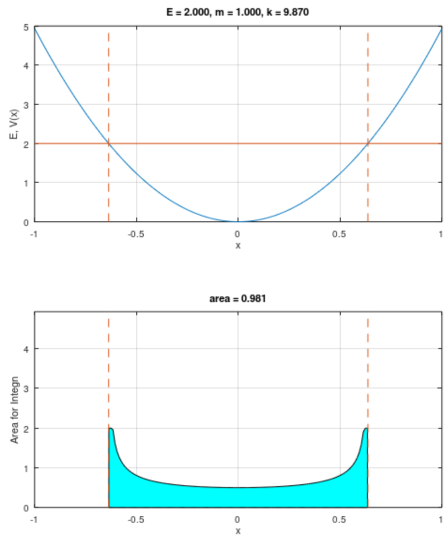



Figure 17. Quadratic potential (top) and associated integral shown as the shaded area (bottom).

The integral is calculated from when the ball starts on the left (at $x \approx -0.64$) and ends up on the right (at $x \approx 0.64$) and has a value of (at $t \approx 0.98$s), this is calculated numerically. This motion of the ball is just one half the period of oscillation, so the period is $T = 2 \times 0.98 = 1.96$s. This system had the coefficients $m = 1$ and $k = 9.87$ which would result in a period $T = 2\pi\sqrt{1/9.87} = 2$s so we see agreement is good.

So we see how to calculate period $T$ from the energy curve

$$T = 2\int\sqrt{\frac{m}{2(E-V)}}dx, \qquad (69)$$

where we integrate between the x-extrema of the position.

### 1.5.4 Why the Quadratic Potential is Special

Let's use the energy integral to find the period of an oscillator with $V(x) = \frac{1}{2}kx^2$. Often nasty integrals like the energy integral can be conquered by substitution, which we shall do here, using $E = \frac{1}{2}kx_0^2$, then eq.68 becomes

$$t = \sqrt{\frac{m}{k}}\int\frac{1}{\sqrt{(x_0^2 - x^2)}}dx \qquad (70)$$

No we can write $x = x_0 \sin\theta$ and using $dx = x_0 \cos\theta \, d\theta$ we have

$$t = \sqrt{\frac{m}{k}} \int \frac{1}{\sqrt{\cos^2\theta}} x_0 \cos\theta \, d\theta \qquad (71)$$

$$= \sqrt{\frac{m}{k}} \theta + t_0,$$

where $t_0$ is a constant of integration. Substituting for $\theta$ we have

$$t = \sqrt{\frac{m}{k}} \sin^{-1}(x/x_0) + t_0, \quad (72)$$

and we end up with the solution

$$x = x_0 \sin(\omega(t - t_0))$$

$$= \sqrt{\frac{2E}{k}} \sin(\omega(t - t_0)), \qquad (73)$$

where $\omega = \sqrt{k/m}$ is our familiar oscillation frequency. The main point to note is that the total energy *E* does not contribute to the frequency term, only to the oscillation amplitude. That's the special thing about a quadratic potential; the frequency of oscillation is independent of the amplitude.

Now what about the area under the integrand graph we mentioned above. The period is proportional to this area, so it must be the same whatever the value of *E*. Figure.18 shows the area for *E*=2 and *E*=3. For the larger *E* the area is wider but no so high, on average. The area for the smaller *E* is higher but less narrow. So they are equal.

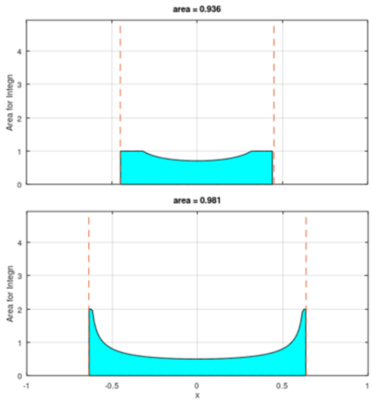



Figure 18. Energy integrals for potential energies 2 (upper) and 3 (lower). The higher energy integral is wider but less tall; both areas turn out to be equal.

## 1.5.5 How to deal with a General Potential Function

We have already seen how to do this; eq.69 shows us how to, in theory, calculate the period of oscillation if we know how the potential $V(x)$ varies with $x$. So how do we do this *in practice*? Well, we resort to a *numerical* integration of the integral. This will always work, but we need to pay attention to the limits of integration where $E - V = 0$, so we need to take numerical care.

One situation which recurs across a number of areas of physics is where the potential has the form shown in Fig.19.

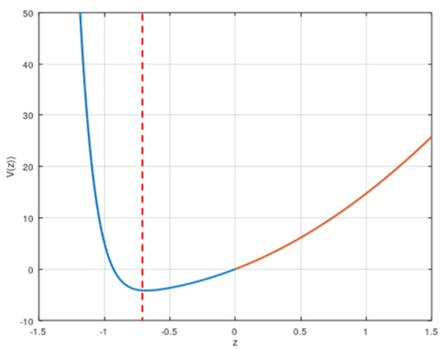



Figure 19. A typical potential plot. The red line is quadratic, and the blue is power-law. A mass will spend most of its time to the right of equilibrium (dashed red line), so its period is dominated by the quadratic part of the potential curve.

The potential has a minimum around $x$ = -0.7, (shown by the red dashed line); to the right of this, the potential is quadratic, but to the left it rises as a power law $\sim x^n$. Think of a ball moving on such a potential curve, how does it behave? When the ball is to the right of the red line it experiences a quadratic potential, so all of our arguments above apply. It will spend a significant time here. But to the left of the dotted line, the potential rises rapidly (so the ball experiences a very large restoring force), so it will spend a small time in this region. We conclude that its period will be dominated by the quadratic region, and this will be approximately one half of the period calculated from eq.69.

### 1.5.6 Oscillations around a General Potential Minimum

Let’s return to the idea of *equilibrium* and we assume we are talking about a *stable* equilibrium. This is a point $x_0$ where the force is zero, $F(x_0) = 0$. Since the force is given by the negative of the derivative of the potential, we have $dV(x_0)/dx = 0$.

Consider the situation shown in Figure 20 where we have a local potential minimum at $x_0$, and consider a small disturbance $\xi$ from this point.

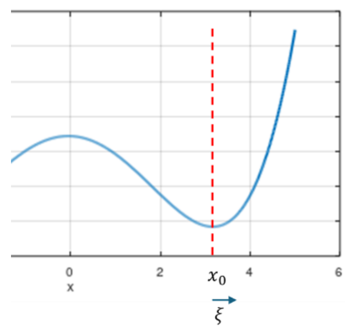



Figure 20. Displacement $\xi$ from a local potential minimum.

We can expand the potential as our usual Taylor series

$$V(x_0 + \xi) = V(x_0) + V'(x_0)\xi + O(\xi^2). \qquad (74)$$

We need the derivative of $V$ to construct the equation of motion which is

$$V'(x_0 + \xi) = V'(x_0) + V''(x_0)\xi + O(\xi^2). \qquad (75)$$

Since $V'(x_0) = 0$ we have

$$m\ddot{\xi} = -V''(x_0)\xi, \qquad (76)$$

which has a harmonic solution with frequency

$$\omega = \sqrt{\frac{V''(x_0)}{m}}. \qquad (77)$$

This is a very useful expression to calculate the frequency of small-amplitude oscillations around a potential minimum. Of course it assumes that $V''(x_0) \neq 0$; if that is not the case, then further terms in the series will need to be considered.

**Example.** Take our mass on a spring with $V(x) = \frac{1}{2}kx^2$ and we wish to find the oscillation frequency around *x* =0. We see that $V'(0) = 0$ and that $V''(0) = k$ so our frequency is $\omega = \sqrt{k/m}$ as we very well know.

# 1.6 The Phase Plane

## 1.6.1 The Concept of the Phase Plane

We know that oscillating systems are modelled as 2$^{nd}$-order ODEs which means that there are two variables, position and velocity, that change with time. For each system we can therefore plot two curves, one of position against time, and the other velocity against time. At each moment in time, the oscillator can be described as vector (position, velocity). So we could plot this vector (point) on axes comprising position and velocity; look at Fig21. At time $t_1$

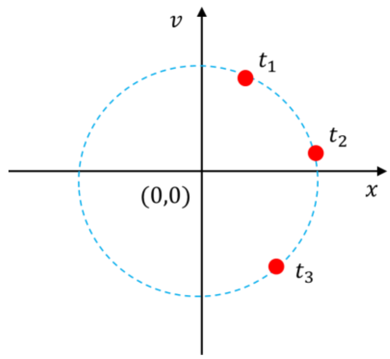



Figure 21. Plot of velocity vs. displacement for an oscillator. Values at three times are shown.

the mass has a small positive displacement and a large positive velocity, so its displacement will increase. Later at $t_2$ the displacement has indeed increased, but its velocity has decreased. At time $t_3$ the velocity is negative, the mass has turned around and is coming back, and its displacement has reduced. Note that we have suggestively added a dashed circle and this suggests that the oscillator is moving around this circle. In fact, if we plotted points at very small time intervals then we would end up with a nice curve, and this would be a circle.

Such an *x-v* plane is known as the *phase plane* and is of great use in describing and understanding oscillations. But the real beauty of the phase plane is that it can easily be applied to non-linear oscillators as we shall see in section 1.6.4.

## 1.6.2 The Phase Plane for our undamped oscillator

We already know how to construct the phase plane for our oscillator, since we know its position as a function of time. So each point on the phase plane can be plotted from

$$x(t) = A\cos\omega t \qquad v(t) = -\omega A\sin\omega t. \qquad (78)$$

In other words, these points lie on an *ellipse* as shown in Fig.22. They do not lie on a circle, since there is an additional factor $\omega$ in the expression for $v$.

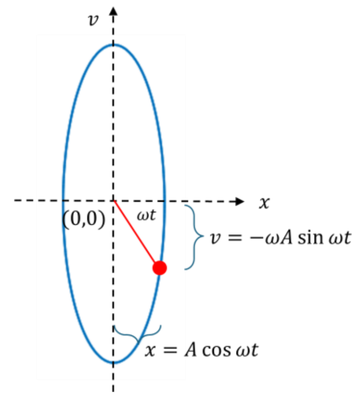



Figure 22. Plot of eq.78 for a SHO. The trajectory is an ellipse.

Now it would be nice if we could arrange things to get the trajectory to be a circle. Why? Well, the changing radius of the ellipse as we travel around does not really give us any real-time information about how the oscillations may develop in time; it is simply a consequence of scaling by $\omega$. Perhaps in more complicated situations the radius might be able to give us some interesting information, after all, simple harmonic motion is the simplest possible.

It's easy to arrange this, all we need to do is to scale the velocity by $\omega$ so that we have $v = \omega v'$ where $v'$ is our new rescaled velocity. Then we can replace eq.78 with

$$x(t) = A\cos\omega t \qquad v'(t)\,\frac{v(t)}{\omega} = -A\sin\omega t, \qquad (79)$$

and so we end up with the trajectory shown in Fig.23, a nice circle.

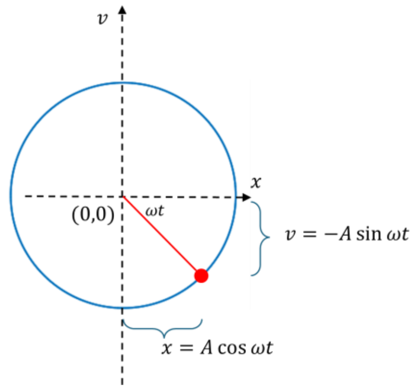



Figure 23. Trajectory when the velocity is scaled according to eq.79.

As time increases so does $\omega t$ and so the red point rotates clockwise.

### 1.6.3 From a 2nd-order ODE to the Phase Plane

We hope you enjoyed the discussion of the phase plane above, but this is not how phase planes are used. In the above, we started with the *solution* $x(t) = A\cos\omega t$, but in most cases we do not know the solution; instead we have the ODEs for the oscillator problem. Here we shall see how we can construct a phase plane starting from the ODEs and that might help us find an analytical solution.

The simple harmonic oscillator is described by a 2nd-order differential equation in *x*, but to make a phase plane we need information about how both *x* and *v* change with time, so we need a 1st-order equation for *x*, and a second 1st-order equation for *v*. Here's how we get them. Let's start by unpicking our 2nd-order equation.

$$\frac{d^2x}{dt^2} = -\omega_0^2 x, \quad i.e., \quad \frac{d}{dt}\left(\frac{dx}{dt}\right) = -\omega_0^2 x,$$

i.e.,

$$\frac{dv}{dt} = -\omega_0^2 x, \quad \text{where} \quad \frac{dx}{dt} = v. \qquad (80)$$

So we now have two 1st-order ODEs which we could use to construct the phase plane. But we won't since they are not symmetrical; the equation for *v* is scaled by $\omega_0^2$ and we would like to remove this asymmetry. You have seen this before when we converted the ellipse into a circle by *rescaling* velocity, so let's try it here writing $v = \omega_0 v'$, and substitution gives us immediately

$$\frac{dv'}{dt} = -\omega_0 x, \quad \text{and} \quad \frac{dx}{dt} = \omega_0 v'. \qquad (81)$$

Now we can see how to construct a phase plane. Ultimately this will be programmed, but let's do it by hand (that'll explain how to code the algorithm). We need to see what's happening at a small number of points on the phase plane, e.g., $(x,v)$ = (1,1), (1,-1), (-1, -1) and (-1,1). We can plug these coordinates into eq.81 and look at the derivatives we get back. Let's take $\omega_0 = 1$; so we build up a table like this:

| *x* | *v* | *dx/dt* | *dv/dt* |
|---|---|---|---|
| 1 | 1 | 1 | -1 |
| 1 | -1 | -1 | -1 |
| -1 | -1 | -1 | 1 |
| -1 | 1 | 1 | 1 |

We use the derivatives as follows. At each point ($x$, $v$) we create and plot a little vector $(\Delta x, \Delta v)$ using the idea that $\Delta x \approx (dx/dt)\Delta t$ and the same for $v$. So, using the same $\Delta t$ we can get the relative sizes of $\Delta x$ and $\Delta v$ and so draw our vectors. Fig.24 shows the results for our tabulated values.

You can see that all the arrows have the same length, and it's not hard to see that they define a circular trajectory  as we have seen, and moreover the direction is clockwise as we saw above. So we have reproduced the phase diagram we saw, but starting from the ODE description of the oscillator and not from the solution!

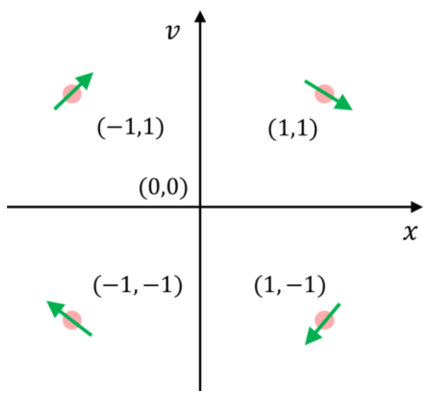



Figure 24. Phase plane showing $dv/dx$ vectors at four points. This shows a clockwise circulation on the plane.

Let's think about the values of $(x,v)$ we chose. It's not hard to see that if we had used (0.5,0.5) etc., then we would have obtained a circle with a smaller radius. So it looks as though we are finding solutions to the ODEs for different initial conditions, and the result is solutions with different amplitudes.

But it gets better. We calculated the vectors at just 4 points in the phase plane, but why not over the whole phase plane, (well at least at a grid of points). Then we would get a *vector field* as shown in Fig.25. The blue arrows show the field and the red circle a trajectory for initial conditions $(x,v) = (1,0)$, point 1, or (0,2), point 3. You can see that starting at point 1 or 2 would result in a circle of smaller radius, corresponding to oscillations of lower amplitude.

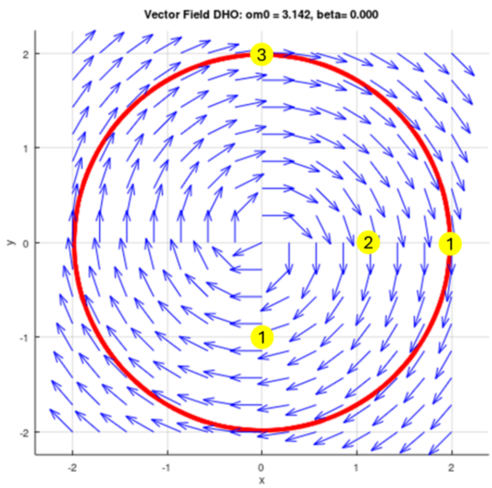



Figure 25. Vector field corresponding to eq.81, trajectory (1) shown in red, see text for other starting points.

Finally, let's have a look at the trajectories when we have included damping. Now the equations, with velocity rescaled, look like this,

$$\frac{dv'}{dt} = -\omega_0 x - 2\beta v', \quad \text{and} \quad \frac{dx}{dt} = \omega_0 v', \qquad (82)$$

and here's the vector field for $\beta = 0.5$. You can see how this really captures the behaviour of the damped oscillator, the ICs are $(x, v) = (2,0)$ and we rotate along a spiral curve as both $x$ and $v$ get smaller with time due to the damping. Finally, we arrive at (0,0), no displacement, no velocity, no bambino, Fig.26.

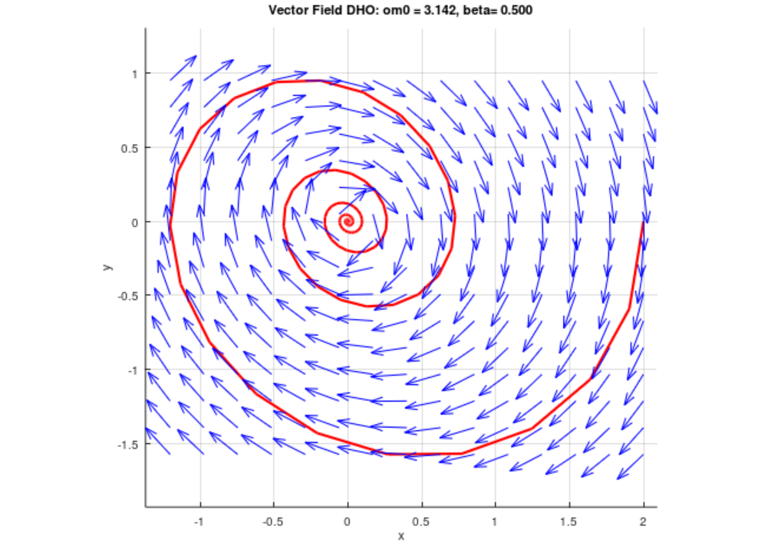



Figure 26. Phase plane and example trajectory for eq.82.

## 1.6.4 Phase Plane for a non-linear oscillator

Just for fun, let's have a look at a non-linear oscillator and discover what the phase plane can tell us about its phenomena. We shall have a closer look at non-linear oscillators in Chapter??, but I say again the following discussion is just for fun.

The Van der Pol oscillator was conceived to model oscillations in an electronic circuit using vacuum tubes, long before transistors were invented; Balthasar van der Pol was a Dutch engineer who did this work in the 1920s. It was later used to model the human heart.

Here it is expressed as our usual pair of 1st-order ODEs, where $e$ is the only parameter which is positive.

$$\frac{dx}{dt} = v', \qquad \frac{dv'}{dt} = -x - e(x^2 - 1)v'. \qquad (83)$$

You will recognize that this is based on our SHO, the only difference is in the 2nd term on the right of the second ODE, and this term is interesting.

The term $-e(x^2 - 1)v'$, contains the factor in brackets that multiplies the velocity $v'$, and we know that a term in $v'$ means damping. But look at the bracketed term; when $x > 1$ then this is positive, so the whole damping term is negative; it is really damping. But when $x < 1$ then the damping term is positive, so it actually is anti-damping pushing energy into the system. So oscillations of initially small amplitude should grow, and oscillations of initially large amplitudes should shrink. Let's see what happens; glance at Figure 27.

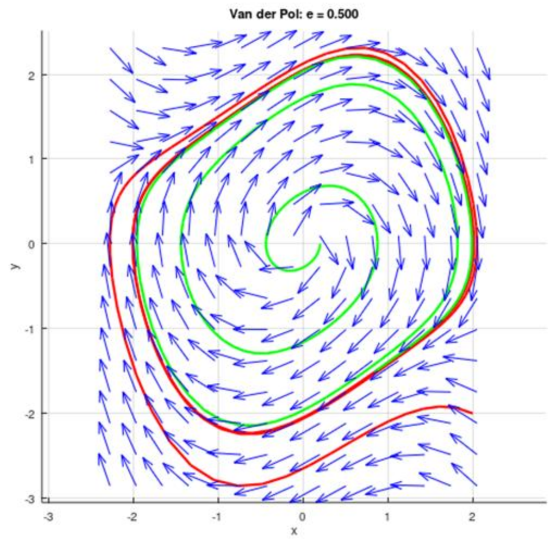



Figure 27. Phase plane and two trajectories for the Van der Pol oscillator, eq.83.

The red trajectory is shown for ICs ($x$, $v$) = (2, -2) so $x > 1$, and the curve spirals inwards but gets stuck in a curve (which resembles a rounded, tilted rectangle). The green trajectory starts at (0.2, 0) so $x < 1$, and the curve spirals outwards until it gets stuck in that same curve. This curve is called a *limit cycle*, and you can just about see that curves from all ICs will end up in this limit cycle. This is a fantastic and unique property of non-linear oscillators, which kept our vacuum tube radios working, and still keeps our hearts beating. Wherever the oscillator starts on the phase plane, it will always be *attracted* to and end up on the limit cycle. If at any time it is on the limit cycle, and suffers a small jolt which takes it off the limit cycle, then it will go back. Such a limit cycle is called an *attractor*.

# 1.7 The Complex Number Approach

## 1.7.1 The Rotating Vector Representation

One demonstration you may have seen is the light projection of a peg on a rotating disc onto a screen. Fig.28 shows (a) the experiment in PhysLab and (b) the corresponding diagram.

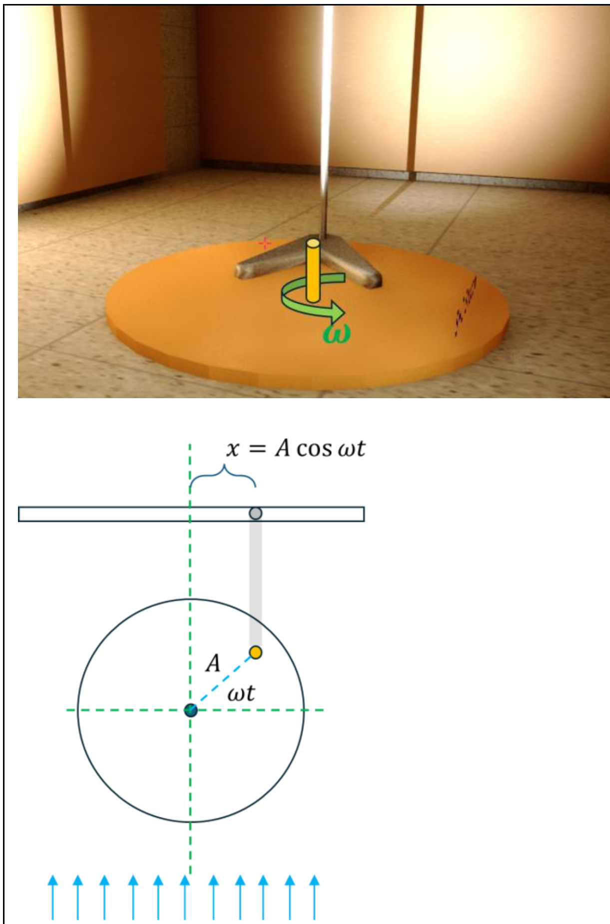



Figure 28. Projection of circular motion on a plane produces the displacement shadow of a harmonic oscillator.

So, as the peg rotates with angular velocity $\omega t$, the parallel light casts a shadow of the peg at position $x(t) = A\cos\omega t$ on the screen. Of course, if we had another light shining in the orthogonal direction, then we would get a shadow at $y(t) = A\sin\omega t$ on the other wall (just visible in the above image). We don't need this additional shadow to get our SHM expression, so we discard it. Nevertheless, what this demonstration does show is an intimate connection between circular motion and SHM. Circular motion contains some extra information we don't need - the sin shadow.

The essence of the above is that SHM can be considered as using the number pair $(A, \omega t)$, and then taking the cos part and discarding the sin part. Wouldn't it be nice if we could write this number pair as a single number $\hat{z}$ (of a different type) and use this in our calculations. Such numbers are called *complex* which we now need to understand.

### 1.7.2 Vectors and Two special rotations

Let's consider a point on a circle with radius *r* and angle *θ* i.e., with polar coordinates $(r, \theta)$, Fig.29.

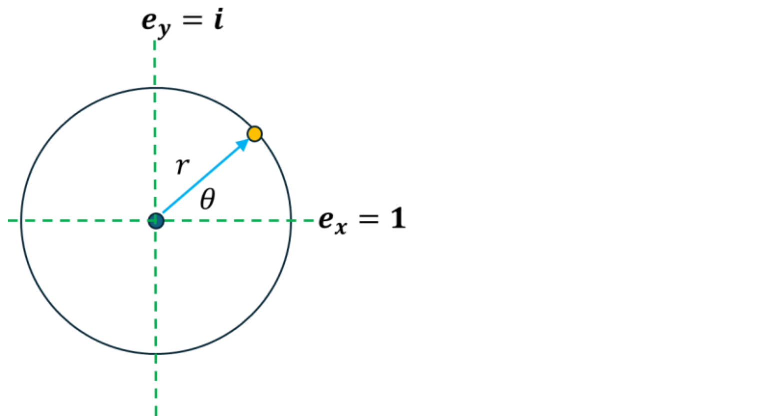



Figure 29.

Using the *basis* vectors $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ (these are vectors of unit length defining the directions of the *x* and *y* axes, and all vectors are bold) we can express this point as a vector

$$\boldsymbol{r} = \boldsymbol{e_x} \cos\theta + \boldsymbol{e_x} \sin\theta\,, \qquad (84)$$

or changing notation, setting $\boldsymbol{e_x} = \mathbf{1}$ and $\boldsymbol{e_y} = \boldsymbol{i}$

$$\boldsymbol{r} = \mathbf{1}\cos\theta + \boldsymbol{i}\sin\theta, \qquad (85)$$

and dropping the **1** gives the result

$$\boldsymbol{r} = \cos\theta + \boldsymbol{i}\sin\theta\,. \qquad (86)$$

So, all we've done here is a bit of vector maths. Now let's take a vector $\boldsymbol{r} = \mathbf{1}$ and let's rotate the vector by 90° anti-clockwise;

Fig.30(a) and (b) shows this. The vectors before and after rotation are

$$\boldsymbol{r} = \mathbf{1}, \quad \boldsymbol{r}' = \boldsymbol{i}, \qquad (87)$$

so we see this rotation is equivalent to multiplying ***r*** by ***i***.

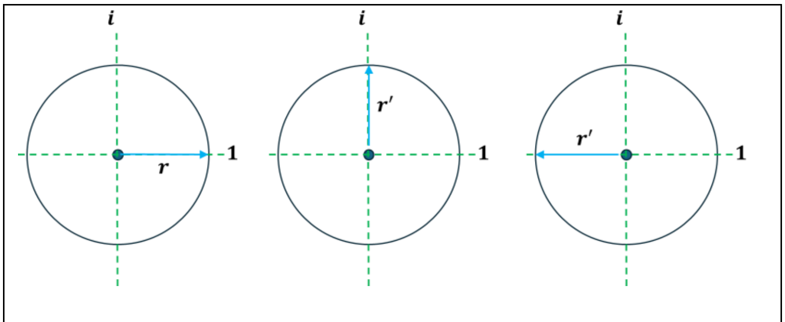



Figure 30.

Now let's rotate by 180° Fig.30(c). Here we have

$$\boldsymbol{r} = \mathbf{1}, \quad \boldsymbol{r}' = -\mathbf{1}, \qquad (88)$$

which is equivalent to multiplying ***r*** by **-1**. But a 180° rotation is just two successive 90° rotations, so we have

$$-1\boldsymbol{r} = i^2\boldsymbol{r}, \qquad (89)$$

from which we find

$$i = \sqrt{-1},$$

which is a pure *imaginary* number. Perhaps you are sceptical, so let's take the general vector in Fig.31 and rotate it by 90°.

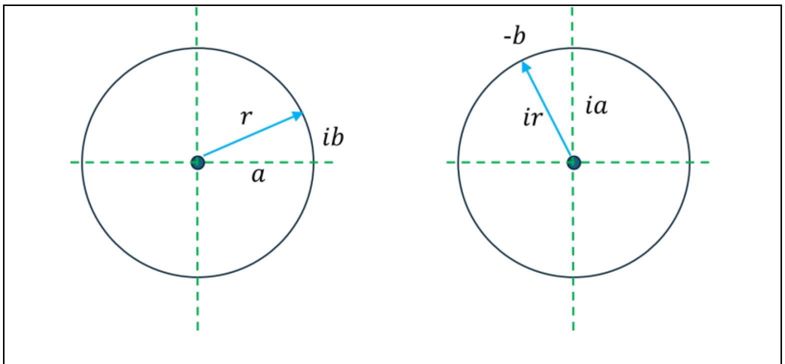



Figure 31.

So we start off with (I'll stop the bold stuff)

$$r = a + ib,$$

and we rotate 90° anti-clockwise

$$ir = i(a + ib) = ia + i^2 b = ia - b,$$

which you can see is correct.

Now let's pause to think about what we have been doing here from a higher view. We have been manipulating *algebraic expressions*, and we have found a link with their *geometrical results*. So, we are suggesting that there is a magical relationship between *algebra* and *geometry*. We shall explore this further in the next section. Also, an expression of the form *a* + *ib* is a mix of a real number *a* and an imaginary number *ib*; such a mix is called a *complex number*. When we shall use complex numbers in various expressions and derivations, we shall end up with a complex number, and we accept that the imaginary part will have no physical significance, so we shall discard this part. That's just like discarding the sin shadow in the demonstration experiment we started with. I hope you see how things hang together.

### 1.7.3 The Complex Exponential

In the above discussions, we have drawn diagrams where a point can be represented as a complex number *a* + *ib* but also in terms of its polar coordinates $(r, \theta)$. So we must ask, is there any relationship between the angle $\theta$ and the complex number $\cos\theta + i\sin\theta$? It turns out there is, and this is one of the most beautiful results in mathematics established by Euler in 1747. We state it here, then show why it works.

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (90)$$

Let's tabulate the Taylor expansion of the three terms in eq.90 where we add the terms in the rightmost columns.

| | | | | | | |
|---|---|---|---|---|---|---|
| $e^{i\theta}$ | 1 | $i\theta$ | $\frac{1}{2!}(i\theta)^2$ | $\frac{1}{3!}(i\theta)^3$ | $\frac{1}{4!}(i\theta)^4$ | $\frac{1}{5!}(i\theta)^5$ |
| | 1 | $i\theta$ | $-\frac{1}{2!}\theta^2$ | $-i\frac{1}{3!}\theta^3$ | $\frac{1}{4!}\theta^4$ | $i\frac{1}{5!}\theta^5$ |
| $\cos\theta$ | 1 | | $-\frac{1}{2!}\theta^2$ | | $\frac{1}{4!}\theta^4$ | |
| $i\sin\theta$ | | $i\theta$ | | $-i\frac{1}{3!}\theta^3$ | | $i\frac{1}{5!}\theta^5$ |

We can see how $e^{i\theta}$ expands to alternately give the terms for the $\cos\theta$ and $i\sin\theta$. Nice eh?

Before we apply all of this to our oscillation theory, there is one more beautiful result which we shall need. Consider the complex number[3] $\hat{z} = re^{i\theta}$. Let's see what happens when we differentiate it

$$\frac{d\hat{z}}{d\theta} = ire^{i\theta} = i\hat{z}, \qquad (91)$$

and on the right hand side we recognize $i\hat{z}$ as the complex number $\hat{z}$ having been rotated anti-clockwise by 90°. So algebraic differentiation is equivalent to a geometrical rotation. This will be incredibly useful in our study of oscillations, since through their ODEs we are differentiating all the time! Again complex numbers bridge algebra and geometry.

### 1.7.4 Linking with Simple Harmonic Motion

To take another step forward in our understanding of complex numbers, let's apply them to the SHO we have already seen. We have the ODE and a candidate solution

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y, \qquad y(t) = A\cos(\omega t + \varphi). \qquad (92)$$

So our angle $\theta = \omega t + \varphi$ and we look for a complex solution of the form

$$\hat{z} = Ae^{i(\omega t+\varphi)} = Ae^{i\omega t}e^{i\varphi}. \qquad (93)$$

We form the first and second derivatives as usual, and assume $A$ is constant

$$\hat{\dot{z}} = i\omega Ae^{i(\omega t+\varphi)} = i\omega\hat{z}\,, \qquad (94)$$

$$\hat{\ddot{z}} = (i\omega)^2 e^{i(\omega t+\varphi)} = -\omega^2\hat{z}\,, \qquad (95)$$

and we substitute into eq.92 to check all is well

$$-\omega^2\hat{z} = \frac{k}{m}\,\hat{z}\,, \qquad (96)$$

and thankfully, we recover eq.7. If you want to repeat the above development using trig functions and complex numbers, please go

[3] We shall denote a complex number using a hat like this $\hat{z}$ while any real variables will be hat-less.

ahead, but it will take you some effort; this could be a useful exercise in appreciating the power of complex numbers.

So, we have taken the algebraic approach, but we know that there is an associated geometrical representation, so let's look at this, Fig.32. The vectors $\hat{z}$, $\hat{\dot{z}}$, and $\hat{\ddot{z}}$ have been drawn in different colors since they are different quantities with different units. Their sizes have been chosen to reflect $\omega > 1$. Each vector shows a phase advance by $\pi/2$ over its predecessor. As time runs on you must see the group of vectors rotating in an anti-clockwise direction but

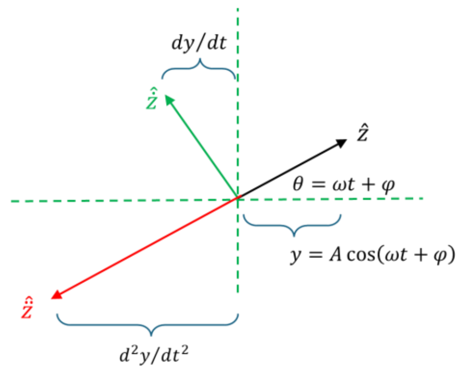



Figure 32. Vectors: displacement (black), velocity (green) and acceleration (red). Phase advances are clearly $\pi/2$.

maintaining the same phase difference between each. Also shown are the projections (shadows) of the complex quantities on the real (horizontal) axis; these are the observables. If you wish to calculate these, please go ahead,

$$y(t) = A\cos(\omega t + \varphi),$$

$$\dot{y}(t) = -\omega A\sin(\omega t + \varphi),$$

$$\ddot{y}(t) = -\omega^2 A\cos(\omega t + \varphi). \qquad (97)$$

You might be able to see the phase difference between these equations, but I think you'll agree this is more explicit using complex numbers and their geometrical representation.

## 1.8 Damped Harmonic Motion

### 1.8.1 A Straightforward Approach

Here's our first major application of complex numbers to a serious problem. The ODE for damped harmonic motion using complex variables is just

$$\ddot{\hat{z}} + 2\beta\dot{\hat{z}} + \omega_0^2\hat{z} = 0, \qquad (98)$$

where $\omega_0^2 = k/m$ is the natural frequency of undamped oscillations of the mass. We then take a guess at a solution and write down its first and second derivates (multiplying by $i\omega$). Note that in our guess we use a different angular frequency $\omega$ since we must not assume the damped oscillation frequency is $\omega_0$. We have two arbitrary coefficients in our guess, $A$ and $\varphi$.

$$\hat{z} = Ae^{i(\omega t+\varphi)},$$

$$\dot{\hat{z}} = i\omega\hat{z}\,,$$

$$\ddot{\hat{z}} = -\omega^2\hat{z}, \qquad (99)$$

and substituting into eq.98 we find

$$-\omega^2 + 2i\beta\omega + \omega_0^2 = 0. \qquad (100)$$

Now this is a quadratic equation in unknown $\omega$ which we can easily solve, but we must be careful, since eq.100 as it stands cannot be solved since we have two real terms and one imaginary term. There is nothing for the imaginary term to cancel with (to give 0), so we conclude that $\omega$ must be a *complex* frequency, whatever that may mean. We should have written the second line of eq.99 as $\dot{\hat{z}} = i\hat{\omega}\hat{z}$. We therefore express $\omega$ as a sum of a real part and an imaginary part

$$\hat{\omega} = \omega_R + i\omega_I, \qquad (101)$$

where both $\omega_R$ and $\omega_I$ are real. Substituting into eq.100

$$(\omega_R + i\omega_I)^2 + 2i\beta(\omega_R + i\omega_I) + \omega_0^2 = 0,$$

$$-\omega_R^2 + \omega_I^2 - 2i\omega_R\omega_I + 2i\beta\omega_R - 2\beta\omega_I + \omega_0^2 = 0. \quad (102)$$

Collecting real and imaginary parts

$$Re\!: \quad -\omega_R^2 + \omega_I^2 - 2\beta\omega_I + \omega_0^2 = 0,$$

$$Im\!: \quad \omega_R\omega_I = \beta\omega_R, \qquad (103)$$

which leads to

$$\omega_I = \beta,$$

$$\omega_R^2 = \omega_0^2 - \beta^2. \quad (104)$$

Now let's bring down our starting guess and insert the expression for complex frequency $\hat{\omega}$

$$\hat{z} = Ae^{i(\hat{\omega}t+\varphi)} = Ae^{-\omega_I t}e^{i(\omega_R t+\varphi)}, \quad (105)$$

and we see that the imaginary frequency part has specified a *damping* term and the real part is a usual harmonic frequency. Finally inserting eq.104 we end up with

$$\hat{z} = Ae^{-\beta t}e^{i\left(\sqrt{\omega_0^2-\beta^2}t+\varphi\right)}, \quad (106)$$

so damping has *reduced* the oscillation frequency from the undamped free value $\omega_0$ to the smaller value

$$\omega = \sqrt{\omega_0^2 - \beta^2}. \quad (107)$$

Of course we are assuming small damping here with $\beta < \omega_0$; we'll return to this l,ater. But let's copy eq.?? and color code it,

$$\hat{z} = Ae^{-\beta t}e^{i\left(\sqrt{\omega_0^2-\beta^2}t+\varphi\right)} \quad (108)$$

The factor shaded yellow corresponds to a vector rotating around a circle with frequency $\omega$; the red damping factor can be interpreted as a time-varying radius of the same circle. We shall develop this idea in the following section.

Fig.33 shows plots of DHM. Top shows the displacement y(t) together with the decay envelope, and bottom shows the velocity v(t) together with y(t). You can see that the velocity *leads* the displacement; peaks in velocity appear before peaks in displacement. We shall soon see why this happens.

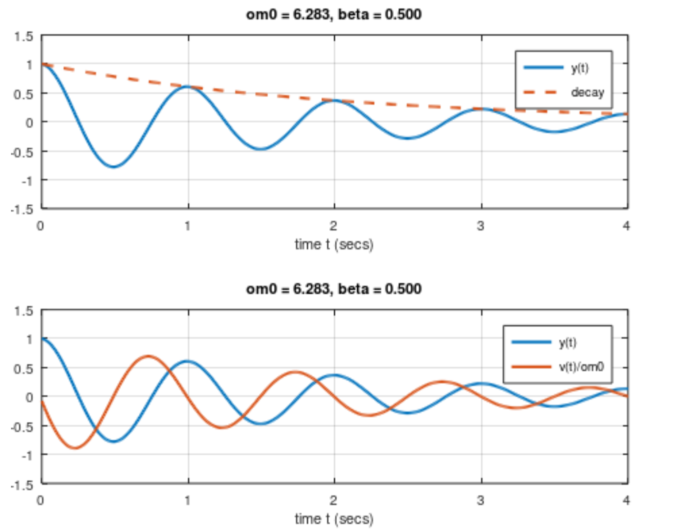



Figure 33. DHM. Top trace shows displacement (blue) with exponential envelope (red dotted). Bottom trace shows displacement and velocity (red).

## 1.8.2 An Alternative Approach

We noted at the end of the last explanation that the complex oscillation $\hat{z}$ could be viewed as a vector rotating around a circle with constant frequency but whose radius was a function of time. So we are thinking of a solution of the form

$$\hat{z} = r(t)e^{i\theta}, \qquad (109)$$

where the real function $r(t)$ plays the role of a time-varying trajectory radius. We have also dropped the phase, as it was not instrumental in the above analysis, and have replaced $\omega t$ by $\theta$; all will become clear!

Now let's consider 'decoupling' the equation into two 1st-order equations, one for $\theta(t)$ and one for $r(t)$. The simplest we can imagine are

$$\dot{\theta} = \omega,$$

$$\dot{r} = -\gamma r, \quad (110)$$

and note we have not used any parameters in the ODE eq.110. Taking the derivatives of eq.109 we find

$$\dot{\hat{z}} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}. \qquad (111)$$

Bringing in our 1st-order guess, eq.109 we find

$$\dot{\hat{z}} = -\gamma r e^{i\theta} + i\omega r e^{i\theta}, \qquad (112)$$

$$\ddot{\hat{z}} = [\gamma^2 - 2\gamma i\omega r - \omega^2 r]e^{i\theta}, \qquad (113)$$

and solving eq.112 for the rightmost term and inserting into eq.113 we have

$$\ddot{\hat{z}} + 2\gamma\dot{\hat{z}} + (\omega^2 + \gamma^2)\hat{z} = . \qquad (114)$$

This has the same structure as eq.98, so we conclude our guess is correct with

$$\gamma = \beta, \qquad \omega^2 = \omega_0^2 - \beta^2. \qquad (115)$$

Inserting these into our two 1st-order ODEs we have

$$\dot{\theta} = \sqrt{\omega_0^2 - \beta^2}, \qquad \dot{r} = -\beta r. \quad (116)$$

Integrating these equations we find

$$\theta(t) = \sqrt{\omega_0^2 - \beta^2}\, t, \qquad r(t) = e^{-\beta t}, \qquad (117)$$

and so eq.109 becomes

$$\hat{z} = r(t)e^{i\theta} = e^{-\beta t}e^{i\sqrt{\omega_0^2-\beta^2}\, t}, \qquad (118)$$

which we have already seen, so everything looks good. The takeaway from this analysis is decoupling the 2nd-order ODE into one 1st-order equation for radius evolution and a second 1st-order equation for angle evolution. Such an approach may be useful in more advanced situations.

### 1.8.3 The Geometrics – Phase Angles

We have emphasized the correspondence between algebra and geometry established using complex numbers. In the development above, we neglected the geometrics because there was so much going on with algebra. So now let's look at vectors and phases. We are particularly interested in the phase difference between displacement and velocity and displacement amplitude.

Let's start afresh with our guess less the arbitrary initial phase.

$$\hat{z} = Ae^{i\hat{\omega}t}.$$

First, we find the velocity

$$\dot{\hat{z}} = i\hat{\omega}\hat{z} = (i\omega_R - \beta)\hat{z}\,, \qquad (119)$$

which tells us how to create the velocity vector from the displacement vector, assuming it is pointing to the right. We draw the vector $i\omega_R\hat{z}$ on the imaginary axis and draw the vector $-\beta\hat{z}$ on the horizontal axis and add these together to give $\dot{\hat{z}}$, shown in Fig.34 (green arrow). We see that the phase angle between displacement and velocity is just

$$\varphi + \frac{\pi}{2}, \quad \text{where} \quad \varphi = \tan^{-1}\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}}, \qquad (120)$$

and we see the velocity *leads* the displacement. If there is no damping then the lead is 90°, and this increases with damping.

To get the angle between acceleration and displacement, we have

$$\ddot{\hat{z}} = -\omega^2\hat{z} = -(\omega_R^2 - \beta^2)\hat{z} - 2i\beta\omega_R\hat{z}\,, \qquad (121)$$

which tells us to draw a vector $-2\beta\omega_R\hat{z}$ on the real axis and $-(\omega_R^2 - \beta^2)\hat{z}$ on the imaginary axis, as shown in Fig.34

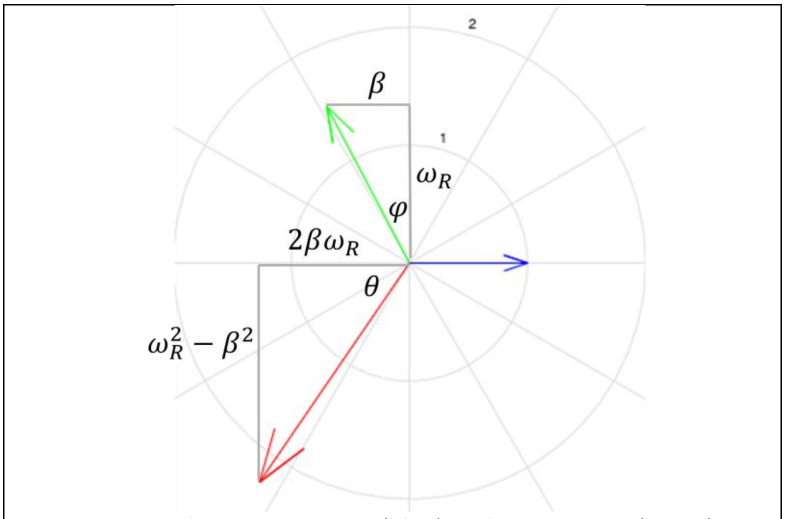



Figure 34. Displacement vector (blue), velocity vector (green) and acceleration vector (red). Phases as calculated in the text.

Following an unwelcome skirmish with algebra, it turns out that $\theta = 2\varphi$ so in Fig.34 the velocity leads the position by $\pi/2 + \varphi$ and the acceleration also leads the velocity by $\pi/2 + \varphi$. As expected, and shown above, both angles reduce to $\pi/2$ when there is no damping.

The takeaway from this is that there is a phase difference between position and velocity, and between velocity and acceleration for a DHO, and these differences increase with damping. So damping does more than progressively reduce the amplitude of oscillation.

## 1.9 Forced Damped Harmonic Motion

Here we shall take our simple mass-spring experiment and add a forcing term; an external force which varies harmonically, something like $F = F_0 \cos \omega t$ where $\omega$ is the frequency of the force which is not the same as the damped frequency of free oscillation of the mass. The resulting behaviour is quite complicated, so we'll build up a discussion in stages; a good place to start is to review some experimental results.

### 1.9.1 Some Experimental Results

The first thing we observe is that the position-time graph shows two stages; there is an initial *transient* state where the oscillation amplitude fluctuates, and a final *steady* state where the amplitude is constant, see Fig.35 You can see that the transient lasts just

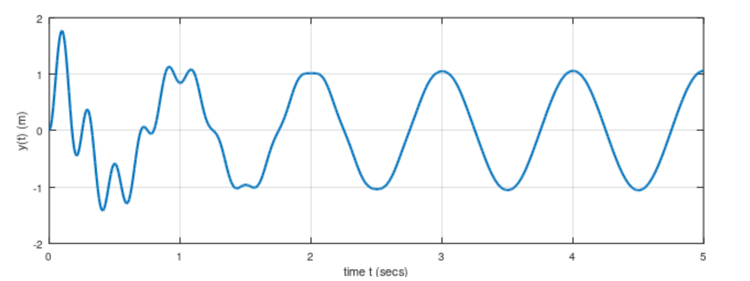


Figure 35. Driven Damped Harmonic Motion. Displacement time trace shows initial transient followed by a stable steady state.

over 2 secs after which we have a steady state. For this experiment the driving frequency is much lower than the natural frequency of oscillation. The transient is dominated by the natural frequency and the steady state by the forcing frequency. This behaviour is hardly surprising since we have a battle between the mass-spring which 'wants' to oscillate at its natural frequency, and the applied force which demands the mass-spring oscillates at the *forcing* frequency. After enough time, the applied force wins the battle.

### 1.9.2 How to force an oscillator

Before we build the ODE describing the motion of the mass, we must decide how the force is to be applied. There are two choices; first we have the situation where the force is applied directly to the mass. This is the easiest to understand. A good example is electric hair-clippers shown exploded in Fig.36. A horizontal arm connected to the clippers is free to oscillate around pivot P. The arm is connected to the casing through a spring, and the force is applied directly to the arm from an electromagnet M.

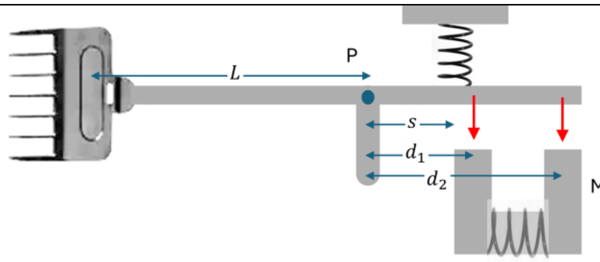



Figure 36. Hair clippers, and example of DDHO.

In this case we could construct the ODE,

$$mL^2\ddot{\theta} + b\dot{\theta} + ks\sin\theta = F(d_1, d_1, \theta)\cos\omega t, \qquad (122)$$

where we are rotating around the pivot point, and the force is a function of the distances shown and also the angle of rotation.

The second excitation method involves an indirect application of the force, usually by displacing one end of the spring. A couple of examples spring to mind. First is a typical lab experiment where a mass is suspended from a spring attached to a vibrator; a transparent cylinder constrains the mass to move vertically Fig.37(a). A second example is a model of earthquakes causing a building to oscillate, Fig.37(b). Here the ground movement $x_g(t)$ stretches the springs which exert a force indirectly on the mass.

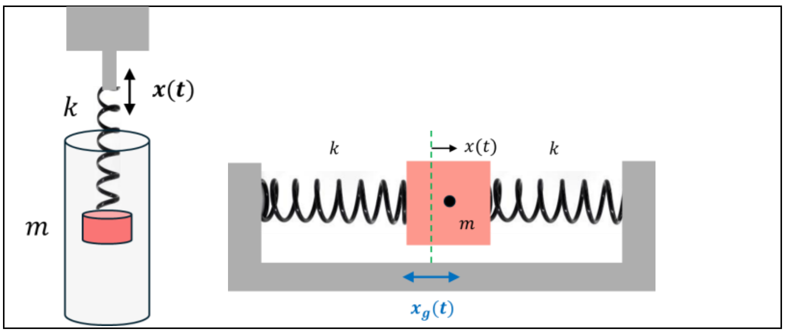


Figure 37. Two examples of indirect force application through a movement *x(t)*.

In this case the ODE becomes

$$m\ddot{x} = -kx - b\dot{x} + kx_g(t), \qquad (123)$$

where $x_g(t)$ could take on a harmonic form.

### 1.9.3 The Steady-state Solution

Let's now apply complex numbers and attempt to solve the ODE expressed in terms of complex variables,

$$\ddot{\hat{z}} + 2\beta\dot{\hat{z}} + \omega_0^2\hat{z} = f_0 e^{i\omega t}, \qquad (124)$$

where $f_0$ is the amplitude of the driving force divided by the mass (so all terms have units of acceleration) and the other symbols are as usual.

For the steady-state solution, the amplitude does not vary, so we need not include a damping term $r(t)$ in the solution. We do expect a phase difference between force and displacement, so we must include this in our guess. We therefore propose:

$$\hat{z} = \hat{C}e^{i\omega t}, \qquad (125)$$

where we have used the forcing frequency $\omega$, (since we are in the steady state where forcing and response frequencies are the same). The complex amplitude $\hat{C}$ is used to contain a possible phase shift

$$(-\omega^2 + 2\beta i\omega + \omega_0^2)\hat{C}e^{i\omega t} = f_0 e^{i\omega t}, \qquad (126)$$

and so we have

$$\hat{C} = \frac{f_0}{(-\omega^2 + 2\beta i\omega + \omega_0^2)}. \qquad (127)$$

Now we can express the complex coefficient $\hat{C}$ in its polar form consisting of a real amplitude $A$ and a real phase $\gamma$

$$\hat{C} = Ae^{-i\gamma}. \qquad (128)$$

We need to solve this for $A$ to figure out how the response amplitude depends on the forcing frequency and system parameters. To do this we multiply eq.128 by its complex conjugate,

$$A^2 = \hat{C}\hat{C}^* = \frac{f_0}{(-\omega^2 + 2\beta i\omega + \omega_0^2)} \frac{f_0}{(-\omega^2 - 2\beta i\omega + \omega_0^2)}$$

$$= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \,. \qquad (129)$$

That's half the work done; before we proceed, let's just plot out curves of amplitude from eq.129 over a range of forcing frequencies for a couple of values of damping, Fig.38. The forcing frequency scale is expressed as multiples of $\omega_0$ and we have also shown the damping $\beta$ as multiples of $\omega_0$.

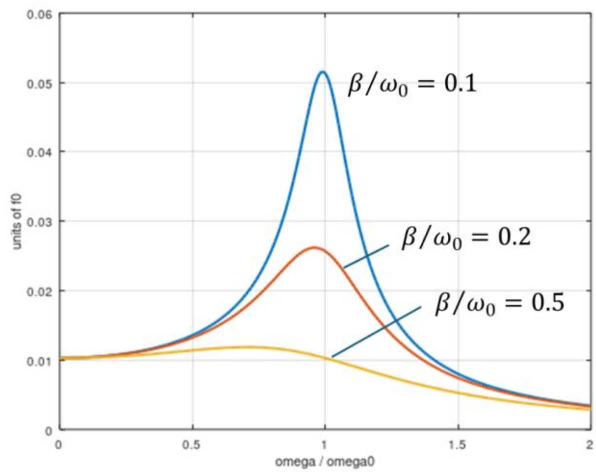



Figure 38. Response curves from eq. 129.

We shall discuss these curves in detail a little later on, but we can note some important characteristics. First, the response amplitude is maximum when the forcing frequency is close to the natural frequency $\omega/\omega_0 \approx 1$. Second, the size of the response decreases as damping is increased, hardly surprising since damping removes energy from the system. Third, the response curve shows a peak with a certain *width* and this width increases with increased damping, so the system is less 'tuned' to the forcing. More on this later.

Now let's return to complete the solution to our system; we still need an expression for the phase $\gamma$ and, more to the point, understand what this means. Remember that $\gamma$ is the phase difference by which the oscillator’s motion lags behind its driving force.

Combining eq.127 and eq.128 gives

$$f_0 e^{i\gamma} = A(\omega_0^2 - \omega^2 + 2\beta i\omega). \qquad (130)$$

Now we invoke the power of complex numbers; we can construct the equivalent geometric form of eq.126. Consider the right-hand side. This tells us to draw a horizontal vector $A(\omega_0^2 - \omega^2)$ which is the real part of the right-hand side and then add on a vertical vector $2A\beta\omega$ which is the imaginary part. Now the left-hand side says draw a vector of length $f_0$ at an angle $\gamma$ to the horizontal, and these two vectors must be identical, since the equation says so. We end up with the vector diagram drawn in Fig.39

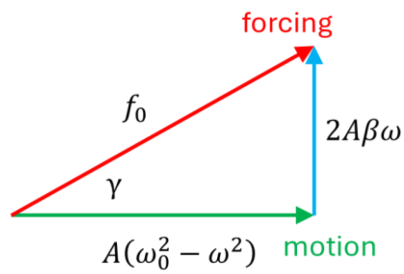



Figure 39. Geometric interpretation of eq.130. Complex forcing vector (red) is sum of real motion vector (green) and imaginary damping (blue).

Now we can easily obtain the phase $\gamma$,

$$\gamma = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right). \qquad (131)$$

So our hunt for a solution to the ODE is now complete. The final stage is to take the real part of

$$\hat{z} = \hat{C} e^{i\omega t} = A e^{-i\gamma} e^{i\omega t},$$

and we end up with

$$x(t) = A\cos(\omega t - \gamma), \qquad (132)$$

where we have expressions for $A$ and $\gamma$. We have already seen how the amplitude depends on forcing frequency; now we need to plot how the phase $\gamma$ depends on this, see Fig.40. Note this phase is a phase *lag* showing that the response motion is delayed from the forcing.

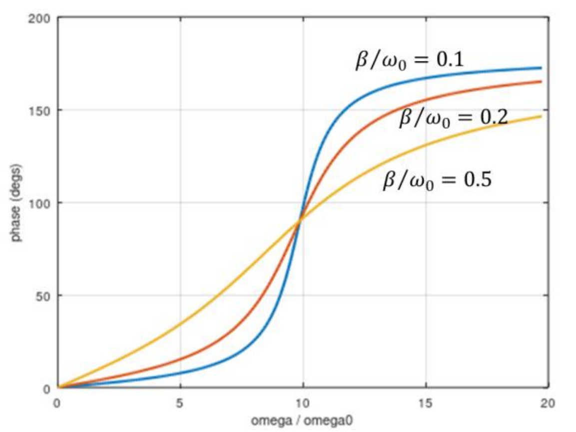



Figure 40. Phase relation from eq.131

The main takeaway here is that the phase increases continually from 0 to 180° as the forcing frequency increases and is exactly 90° when the system is forced at its natural frequency. To make this concrete, consider the behaviour of a can of soup suspended from a long rubber band held in your hand and forced with a frequency lower than $\omega_0$ and then with a frequency above $\omega_0$. If you actually perform this experiment, try to keep the amplitude of your hand constant. The results are in Fig.41.

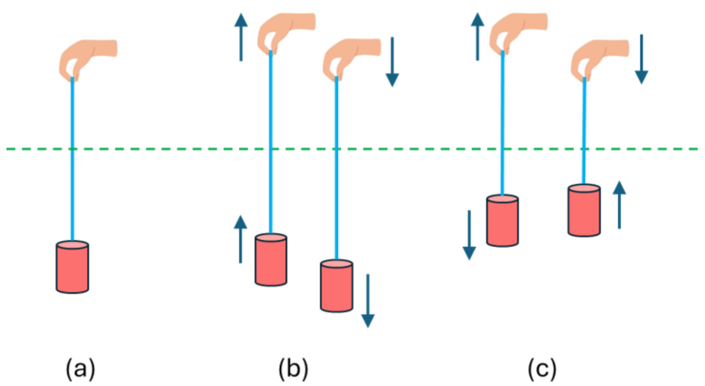



Figure 41. Suspended can of soup, (a) shows it at rest. In (b) the forcing frequency is low, and the can moves (arrows show velocity) in phase with the forcing. In (c) the response is out of phase (motion exaggerated)

You can see for low frequencies, when the hand moves up the can moves up and *vice-versa*, this is response in phase with the forcing. For higher frequencies, when the hand moves up the can moves down, and *vice-versa*, this is out-of-phase response. Note that this is true irrespective of the amount of damping $\beta$. The effect of increasing $\beta$ is to 'smooth out' the rate of change from in-phase to out-of-phase response.

### 1.9.4 The Force Balance Analysis

Let's work with this ODE where we make the experimental coefficients $m$, $k$, and $b$ explicit, but remain using complex numbers.

$$m\ddot{\hat{z}} + b\dot{\hat{z}} + k\hat{z} = F_0 e^{i\omega t}, \qquad (133)$$

where $F_0$ is the amplitude of the driving force. This ODE expresses an equality between various forces within the oscillator and the driving force

$$F_{inertia} + F_{resis} + F_{stiff} = F_{apply}. \qquad (134)$$

The vectors representing these forces are shown on the Argand diagram in Fig.42 for a particular set of coefficients.

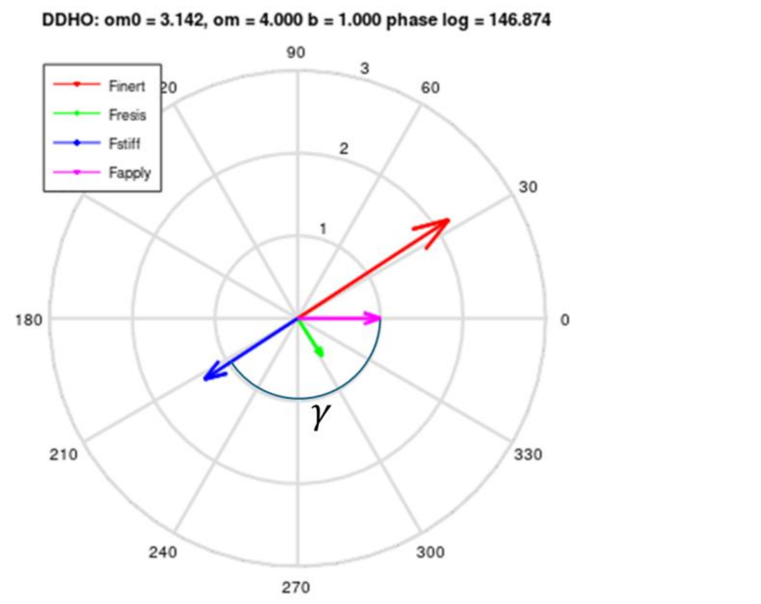



Figure 42. Force balance vectors. Apply (purple) is the sum of inertia (red), resistance (green) and stiffness (blue). Phase angle is shown.

The phase angle $\gamma$is shown; remember this is the phase difference by which the oscillator motion (displacement) lags behind the driving force. We shall consider three special cases where this 'force balance' equation can be simplified, leading to some approximate solutions for the oscillation displacement.

Here we consider three special cases, and for each we assume that damping is relatively small, $\beta \ll \omega$. We shall draw the following forces on the Argand diagram (i) the exciting force, (ii) the force provided by the spring, (iii) the damping force, (iv) the force experienced by the mass.

#### 1.9.4.1 Fast Driving

Here we consider $\omega \gg \omega_0$. Since the driving frequency $\omega$ is high, the inertia force $F_{inertia}$ will be relatively large (since it is proportional to $\omega^2$) and we can ignore $F_{resis}$ and $F_{stiff}$. The force balance then becomes $F_{inertia} = F_{apply}$ and eq.133 simplifies to

$$x(t) = -\frac{(F_0/m)}{\omega^2}\cos\omega t, \qquad (xxa)$$

and we see that $x(t)$ is small. You can see from the vectors in Fig.43 that the driving force is in phase with the acceleration, ($F_{inertia}$) since it provides essentially all of the force. Also, the driving force and the displacement are out of phase, ($\gamma \approx 180°$) which gives the minus sign in eq.135. You can see this in the soup-can sketch Fig.41(c). The force values and the vectors were calculated for $\omega_0 = \pi$ and $\omega = 10$; we have chosen these values so the vectors do not overlap.

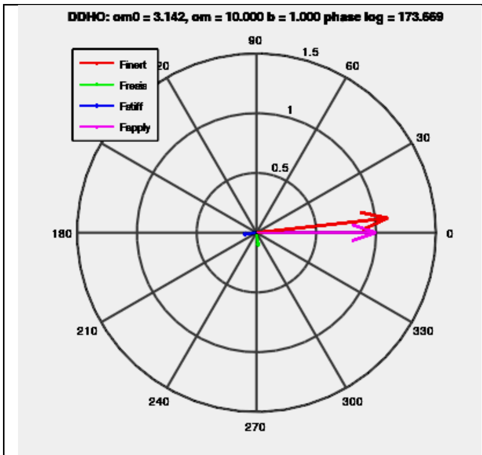



Figure 43. Driving force (purple) in phase with the acceleration (red).

**Force values.**
Finert=1.096 + 0.122i
Fresis=0.01216-0.1096i
Fstiff=-0.108177-0.012i
Fapply = 1

#### 1.9.4.2 Slow Driving

Here we consider $\omega \ll \omega_0$. Here the acceleration and velocity are both relatively small, so we can ignore $F_{inertia}$ and $F_{resis}$ and so the force balance becomes $F_{stiff} = F_{apply}$ and eq.133 simplifies to

$$x = \frac{(F_0/m)}{\omega_0^2} \cos \omega t. \qquad (136)$$

The vector diagram Fig.44 calculated for $\omega_0 = \pi$ and $\omega = 1$ shows that the driving force is now in phase with the displacement, so the displacement just follows the force as shown for soup can in Fig.41(b).

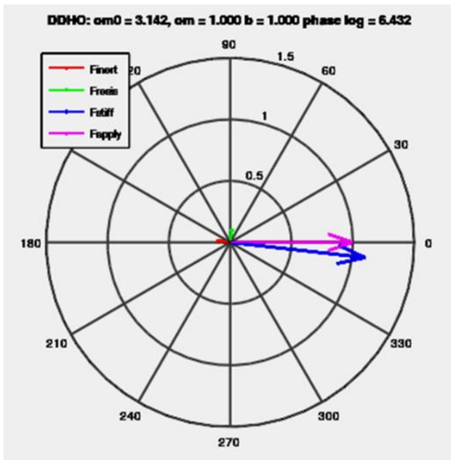



Figure 44. Driving force (purple) in phase with displacement (blue)

**Force values**
Finert = -0.111 +0.013i
Fresis = 0.0126 +0.1113i
Fstiff = 1.099 - 0.1239i
Fapply = 1

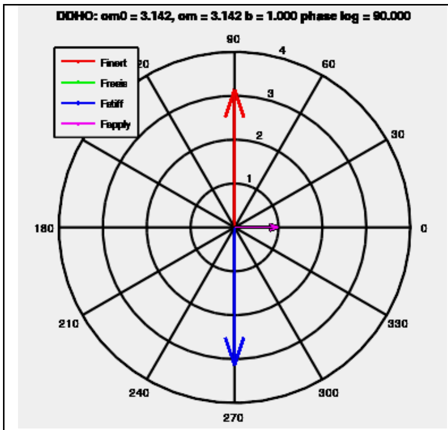
### 1.9.4.3 Driving at Resonance

Here we consider $\omega = \omega_0$. Looking at the vectors and force values in Fig.45 we see that $F_{inertia} + F_{stiff} = 0$, so the stiffness and inertia forces are equal and opposite. The driving force equals the resistance force, and they are in phase $F_{inertia} + F_{apply}$ and are 90° out of phase with the inertia and stiffness forces. Since these force-pairs are orthogonal, they do not interact, and so we effectively have two 'subsystems'; the spring and mass force each other just as if they were performing undamped harmonic motion, and the applied force takes away the damping so this can happen.

Simplifying eq.133 we find

$$x(t) = \frac{(F_0/m)}{2\beta\omega_0} \sin \omega t, \qquad (137)$$

where the 90° phase shift, we mentioned above is apparent. Since driving cancels damping, this explains the peak in the amplitude response curve. Looking at it in another way, the driving force is in phase with the velocity, (green arrow in Fig.45), and since power is force times velocity, the input power is largest when force and velocity are in phase. We shall come onto power shortly.



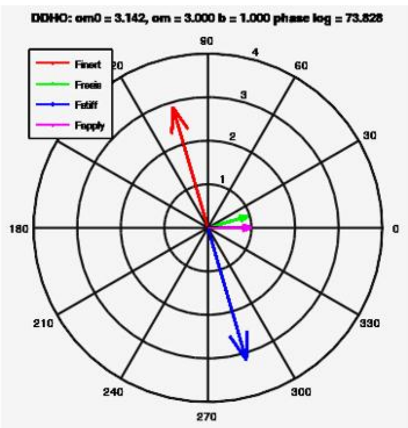




Figure 45. Phases at resonance. Driving force (purple) equals resistance force (green). Acceleration (red) is equal and opposite to the spring force (blue).

**Force values**

Finert = 0 + 3.14i

Fresis = 1.0

Fstiff = 0 - 3.14i

Fapply = 1.0

## 1.9.5 Designing for Resonance

We have seen that the amplitude of the response shows a definite peak when the forcing frequency is close to the natural frequency of the system. It's worth giving this a little more thought. Clearly the amplitude is a maximum when the denominator of eq.129

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 \qquad (138)$$

is a minimum. How can we arrange this? Well it depends. Sometimes you need to tune the system (i.e., change $\omega_0$) when the forcing frequency $\omega$ is known, and fixed. An example could be designing a wave energy convertor which bobs up and down in ocean waves, which have a typical average frequency. Then the minimum clearly occurs when you set $\omega_0 = \omega$. So you choose the equivalent mass and spring constant of the bob. In other situations $\omega_0$ may be fixed and you need to adjust $\omega_0$ for resonance. One example is a paint mixer, where a bucket of paint components is shaken on a table connected with springs. In this case differentiation shows that you get a resonant peak when

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}, \qquad (139)$$

which is close to $\omega_0$ when damping is relatively small. In any case, a rough guideline for the maximum amplitude, from eq.129 is

$$A_{max} \approx \frac{f_0}{2\beta\omega_0}. \qquad (140)$$

### 1.9.6 Difference between DHO and Driven-DHO

There are many differences, but here we summarize a few to avoid confusion since different terms mean different things. **Phase differences**. For a DHO the phase difference between velocity and displacement depends on the amount of damping. For DDHO this phase difference is always 90°.

## 1.10 Forced Oscillations - Powers

So far, we have viewed driving a damped harmonic oscillator as providing an exciting *force* and we were interested to see its response *displacement*. There is another view we can take which is very useful in two broad areas, (i) general engineering applications, (ii) electrical and electronic circuits. Here, we are interested in discussing the *power* supplied to the damped oscillator, and in particular where it ends up within the system components of spring, mass, and damping.

The input power is just the rate of supplying energy to the oscillator; since the work done has the form $Fx$ then the expression for power has the form $P = d(Fx)/dt$ which for a constant force is just $P = Fv$. So now we are interested in the force and *velocity*. You may remember that the power supplied to an electrical circuit has the form $P = VI$ where voltage is analogous to force, and current (which is a flow) is analogous to velocity.

### 1.10.1 The Concepts of Reactance and Impedance

Let's start with our ODE in complex form, but using our physical parameters *m*, *k*, and *b*. As mentioned above, we shall focus on *using velocity* $\dot{\hat{z}}$ *rather than displacement* $\hat{z}$ as our principal variable. So we would like to reformulate our ODE to link force and velocity. Starting with

$$m\ddot{\hat{z}} + b\dot{\hat{z}} + k\hat{z} = f_0 e^{i\omega t}, \qquad (141)$$

we know that velocity $\dot{\hat{z}} = i\omega\hat{z}$ and that acceleration $\ddot{\hat{z}} = i\omega\dot{\hat{z}}$ so the above equation becomes, writing the r.h.s. as $\hat{f}$

$$\left(im\omega + b + \frac{k}{i\omega}\right)\dot{\hat{z}} = \hat{f}. \qquad (142)$$

Denoting the real part of the bracket as *R* and the complex part as *X* we have

$$R = b, \qquad iX = i\left(m\omega - \frac{k}{\omega}\right). \qquad (143)$$

So the sum becomes the complex *impedance* $\hat{Z}$, a term familiar in electricity,

$$\hat{Z} = R + iX, \quad (144)$$

and we call the components of impedance the *resistance* R and *reactance* X. You must understand that resistance is connected with damping, and reaction with the oscillation. As expected, $\hat{Z}$ is a complex quantity and may be expressed as

$$\hat{Z} = |\hat{Z}|e^{i\varphi}, \qquad (145)$$

where $\varphi$ is the phase angle between real and imaginary components of $\widehat{Z}$[4]. This is shown in Fig.46.

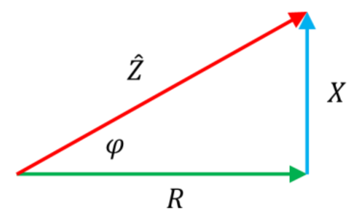



Figure 46. Relation between impedance and its components, resistance and reactance.

The phase angle is given by

$$\varphi = \tan^{-1}\frac{X}{R}. \qquad (146)$$

From eq.142 and eq.144 we obtain our desired relation between force and velocity as

$$\hat{f} = \hat{Z}\hat{u}, \qquad (147)$$

where we will use symbol $\hat{u}$ for the complex response velocity.

Now let us consider the balance of forces within our oscillator. The ODE eq.141 can be interpreted as a statement of this force balance

$$F_{Inertia} + F_{damp} + F_{spring} = F_{applied},$$

$$m\hat{a} + b\hat{u} + k\hat{z} = \hat{F} = \hat{Z}\hat{u}, \qquad (148)$$

or, in terms of $\hat{u}$

$$(mi\omega)\hat{u} + b\hat{u} + \left(\frac{k}{i\omega}\right)\hat{u} = \hat{Z}\hat{u} \qquad (xx)$$

$$X = (m\omega) - \left(\frac{k}{\omega}\right) \qquad R$$

,(149)

where we show which terms lead to the reactance and resistance to give the impedance $\hat{Z} = iX + R$.

[4] This is not the same as all previous uses of this symbol to represent phase (we ran out of good symbols). This is a different phase!

An example is given in Fig.47 for parameters $m$ = 1kg, $k$ = 9.87N/m, $b$ = 1kg/s and forcing frequency $\omega$ = 4. The direction of the applied force is indicated to provide a reference point for this and subsequent vector plots.

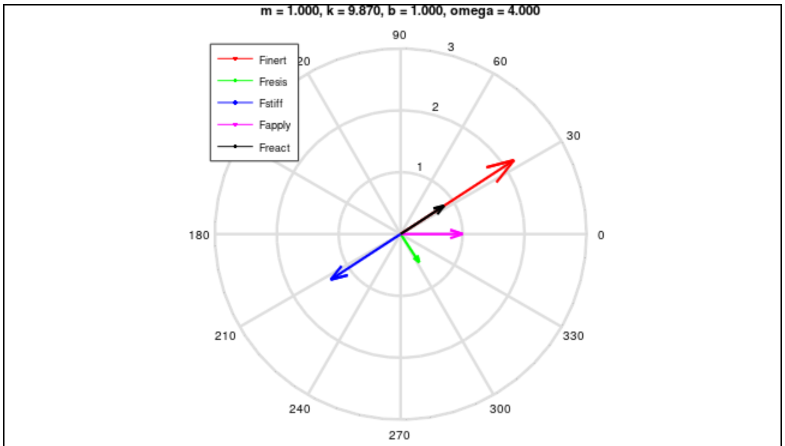



Figure 47. Acceleration (red) and spring force (blue) have phase difference 180°. Black (reactance) and green (resistance) sum to the 'impedance force' (purple) which is the applied force.

Check that you understand that the inertia force $m\hat{a}$ and the spring force $k\hat{z}$ have a phase difference of 180°. You can see that from the expression for reactance in eq.149 where they have opposite signs. Or you could argue from the ODE that $m\hat{a} = -k\hat{z}$, or using complex numbers that $\hat{u} = i\omega\hat{z}$ and $\hat{a} = i\omega\hat{u}$ hence $\hat{a} = -\omega^2\hat{z}$).

Note also the triad of green, black and magenta vectors which represent, respectively, the resistance force $R\hat{u}$ and the reactance force $iX\hat{u}$ which sum to give the 'impedance force' $kZ$ which is of course the applied force $\hat{F}$.

### 1.10.2 Power Delivered and Consumed by the DHO

Here we shall make a distinction between the power *delivered* to the oscillator which will have the form $P = Fv$ (like $P = VI$ for an electrical circuit) and the power *consumed* by the oscillator (by its damping) which will have the form $P = Ru^2$ (like $P = RI^2$ for an electrical circuit.

Let's start with the power *delivered* to our oscillator. Power is a real quantity and given a complex driving force and response velocity then we must calculate

$$P(t) = Re\{\hat{f}\}.Re\{\hat{u}\}\,. \qquad (150)$$

Using complex conjugates to get both real terms on the right we have

$$P(t) = \left(\tfrac{1}{2}\hat{f}e^{i\omega t} + \tfrac{1}{2}\hat{f}^*e^{-i\omega t}\right)\left(\tfrac{1}{2}\hat{u}e^{i\omega t} + \tfrac{1}{2}\hat{u}^*e^{-i\omega t}\right). \qquad (151)$$

Note we have used a complex force and velocity amplitudes to contain any phase difference between force and velocity. Expanding, we get

$$P(t) = \tfrac{1}{4}\left(\hat{f}\hat{u}^* + \hat{f}^*\hat{u}\right) + \tfrac{1}{4}\left(\hat{f}\hat{u}e^{2i\omega t} + \hat{f}^*\hat{u}^*e^{-2i\omega t}\right). \quad (152)$$

This may look a little daunting at first but look for the symmetries and you will see it becomes interesting. The last two terms are complex conjugates, so represent a real harmonic oscillation with frequency $2\omega$, so their average over one period of oscillation is zero. Hence the delivered time average is

$$\overline{P(t)} = \tfrac{1}{4}\hat{f}\hat{u}^* + cc = \tfrac{1}{2}Re\{\hat{f}\hat{u}^*\} \qquad (153)$$

Using the relation $\hat{f} = \hat{Z}\hat{u}$ to replace $\hat{u}^*$ we have

$$\overline{P(t)} = \tfrac{1}{2}Re\left\{\hat{f}\frac{\hat{f}^*}{\hat{Z}^*}\right\} = \tfrac{1}{2}Re\left\{\hat{Z}\frac{\hat{f}\hat{f}^*}{\hat{Z}\hat{Z}^*}\right\} = \tfrac{1}{2}Re\{\hat{Z}\}\frac{\left|\hat{f}\right|^2}{\left|\hat{Z}\right|^2}$$

$$= \frac{1}{2}\frac{R}{R^2 + X^2}\left|\hat{f}\right|^2 \qquad (154)$$

Keep this expression in mind for the moment, since we are about to see it again.

Now let's derive an expression for the power *consumed* by the damping. This is

$$P_{damp} = Ru^2 = R\left(\tfrac{1}{2}\hat{u}e^{i\omega t} + \tfrac{1}{2}\hat{u}^*e^{-i\omega t}\right)^2$$

$$= \tfrac{1}{4}\left(2\hat{u}\hat{u}^* + \hat{u}^2e^{2i\omega t} + \hat{u}^{*2}e^{-2i\omega t}\right), \qquad (155)$$

where the last two terms are harmonic motion with frequency $2\omega$, so their average over one period of oscillation is zero, so the time average *consumed* power is

$$\overline{P_{damp}(t)} = \tfrac{1}{2}R\hat{u}\hat{u}^* = \tfrac{1}{2}R\frac{\hat{f}\hat{f}^*}{\hat{Z}\hat{Z}^*} = \frac{1}{2}\frac{R}{R^2 + X^2}\left|\hat{f}\right|^2 \qquad (156)$$

which is the same as the delivered power eq.154; it has to be. How could we *design* a system so that these powers are maximal? We have two variables $X$ and $R$ which we are free to choose. Since the reactance $X$ appears only in the denominator, then it makes sense to consider $X = 0$. What does this mean for the constituent components of $X$ ($m$ and $k$)? Well from eq.149 we see that the condition

$$\omega^2 = k/m \qquad (157)$$

will do the trick. The maximum power is then $|\hat{f}|^2/2R$. The associated vector diagram is shown in Fig.48 (well, close to resonance so we can see all the arrows). The consequence of setting reactance to zero is that the force and velocity arrows are in phase, so that the force is doing the maximum amount of work on the oscillator, so its amplitude is largest.

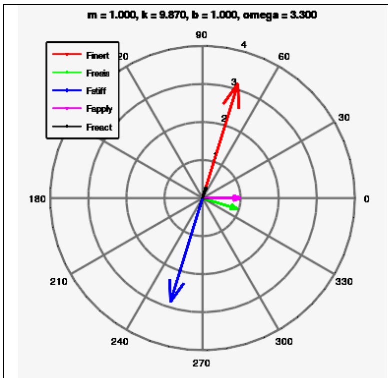


Figure 48. Zero (almost) reactance. Force (purple) and velocity (green) are in phase.

Of course, this choice implies that we are able to specify the value of either $m$ or $k$ in our *engineering design*. This may not be possible; instead we may only be able to specify the size of $R$. Finding the maximum power while varying $R$, we find the condition that $R = X$ and then the maximum power becomes $|\hat{f}|^2/4R$, half what we could achieve by varying $X$. In this case, the vector diagram appears in Fig.49 where the phase between force and velocity is 45°.

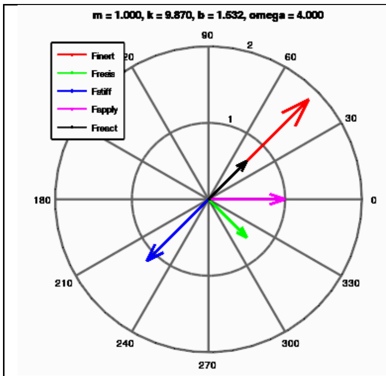



Figure 49. Reactance and resistance magnitudes are equal. Phase difference between force (purple) and velocity (green) is 45°.

In addition to these average powers, there are fluctuations in the delivered and consumed power, and the difference will not be zero, so some power is stored in the oscillating system.

So let's look at power flow over a *single cycle* of excitation. First let's look at the *delivered* power. We take eq.152 and insert the average power for the first two terms, so we now are considering the terms with frequency $2\omega$ which we excluded from our averaging.

$$P(t) = \overline{P(t)} + \left(\tfrac{1}{4}\hat{f}\hat{u}e^{2i\omega t} + cc\right)$$

$$= \overline{P(t)} + \tfrac{1}{2}\left|\hat{f}\hat{u}\right|\cos(2\omega t - \varphi). \qquad (158)$$

Likewise, we have for the *consumed* power

$$P_{damp}(t) = \overline{P_{damp}(t)} + \left(\tfrac{1}{4}R\hat{u}^2 e^{2i\omega t} + cc\right)$$

$$= \tfrac{1}{4}R|\hat{u}|^2\cos(2\omega t - 2\varphi), \qquad (159)$$

and as expected the delivered and consumed power have a different time variation during one cycle; the consumed power has double the phase shift of the delivered power. Why exactly is this? Well, the delivered power used is

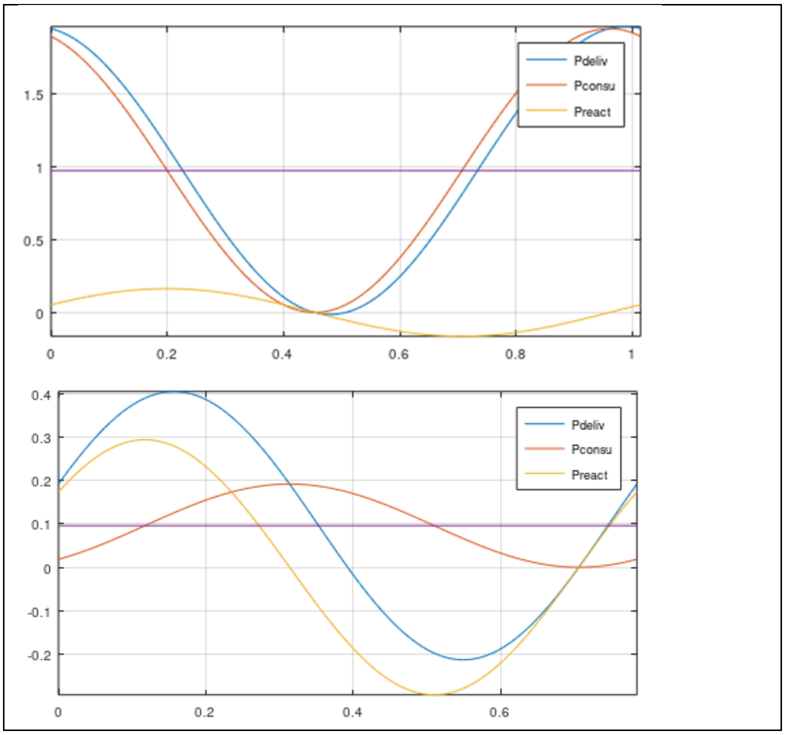
$$\hat{f}\hat{u} = f_0 u_0 e^{-i\varphi}, \qquad (160)$$

and the consumed power used is

$$\hat{u}\hat{u} = u_0 e^{-i\varphi} u_0 e^{-i\varphi} = u_0^2 e^{-2i\varphi}, \qquad (161)$$

so there's that factor 2. Remember the phase shift $\varphi$ is established by the ratio of $X$ to $R$.

Some understanding of power flow can be gleaned from plots of delivered and consumed powers over a single cycle. Fig.50(a) shows a plot for $\omega = 3.1$ and $b = 0.5$, so $X$ is small as is the phase $\varphi$. So we are close to resonance. As expected, the plot shows delivered and consumed power are about equal and close to being in-phase, and there is very little reactive power. So power is being used to fight damping. Also shown is above resonance Fig.50(b) and below resonance Fig.50(c) where delivered and reactive power are in phase, with converted power at 90° phase and relatively small. Here, a significant part of the delivered power is used in making the system oscillate at a frequency which it does not like to oscillate at, (it prefers its natural frequency).



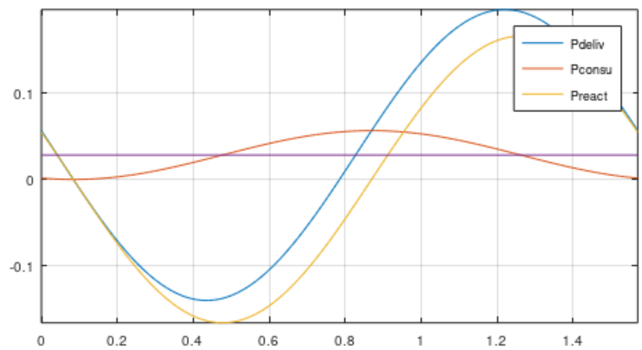



Figure 50. Top, close to resonance, delivered (blue) and consumed power (red) are about equal. Bottom delivered and reactive (orange) power are in phase.

# 1.11 Some Examples

## 1.11.1 Mass suspended from a Rubber Band

You might find the above heading somewhat amusing since we do not often see rubber springs in the world around us, or at least we do not recognize them. Note we are talking about *solid rubber* springs, not rubber being used as a container as in the case of air springs which are familiar to us. We know that rubber is *compliant*, and we make use of flexible rubber hoses, e.g., connecting parts of our vehicle cooling system, or as ‘bushes’ between steering or suspension components. Here their compliance helps them to take up the slack. Also rubber has great potential to absorb vibrations and is used with great effect in vibration analyzers; but here it is not their ‘stiffness’ which is important but their ‘visco-elastic’ properties which make them absorb. But what about real rubber springs? They are all around us; a good example is in train bogies that support carriages, see Fig.51 Such springs are able to support large loads while ensuring smooth rides.


Figure 51. Get others and permissions from Continental

One reason we are studying rubber springs, is that their force-extension graph is very different from a metal coil spring which obeys Hooke's Law, where deflexion is proportional to force *'Ut tension, sic vis'*. This relationship is shown diagrammatically in Fig.52 where there are two clear regions of behaviour.

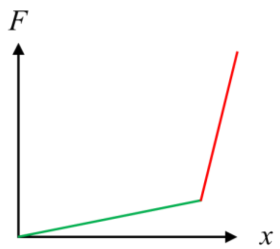



Figure 52. Model of rubber band elasticity. Initial region (green) is not so stiff, final region (red) is very stiff.

Say we perform an experiment where we extend a rubber band spring from zero upwards. At first the force required (or measured) increases quite slowly (green line), but at a certain point, the rate of increase shows a dramatic increase (red line). Since the gradient of the force-extension curve is the spring stiffness $k$ then the rubber band spring has suddenly got much stiffer. How could this happen? Well, consider the molecular structure of a rubber band. Sitting on your desk it comprises many long chains of rubber molecules, all intertwined and mixed up. So when you first stretch a rubber band, you are *uncoiling* these chains which is relatively easy. This takes energy which requires a force as you extend. Suddenly, all the chains are uncoiled, but you keep stretching. So now you are stretching the inter-molecular bonds which are very stiff. So the stiffness dramatically increases.

We wish to model a system comprising a mass attached to a rubber band, Fig.53. The following notes are an overview, see Chapter XX for a detailed discussion.

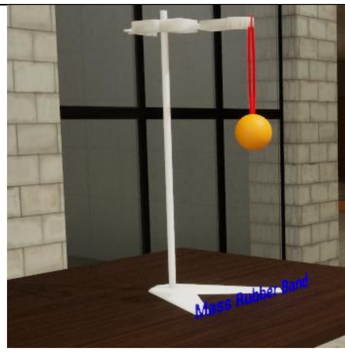



Figure 53. PhysLab mass rubber-band oscillator.

To create our model, we need to set up an equation of motion which will take the form

$$m\ddot{z} = F(z) - \nu\dot{z} - mg, \qquad (163)$$

which is familiar, but we must model the non-linear behaviour of the rubber band through the function $F(z)$ rather than our usual linear term $-kz$.

We are free to choose the details of $F(z)$ as long as it reproduces the behaviour shown in Fig.52. The initial green segment is linear, so for this part we choose a linear term $-kz$. For the red segment we need a function which rises rapidly with $z$, so we choose a term $-kz^n$. This will give us a function for *extension* of the rubber band. For *compression* there is no stretch of molecular bonds, so we do not need this second term. So our actual function has the form

$$F(z) = -kz - \alpha k^9 \qquad z < 0$$

$$= -kz \qquad z > 0\,. \qquad (164)$$

Note that we have chosen $n$ = 9, and also the $z$-axis is oriented upwards, so, loading the rubber band with a mass, produces a negative z.

This function is shown in Fig.54 together with the equivalent potential $V(z)$. Both graphs show the *asymmetry* of the model we have set up. When the rubber band is extended, then it shows non-linear behaviour and when it is compressed, it behaves like a familiar metal spring. Let's look for the equilibrium position of the

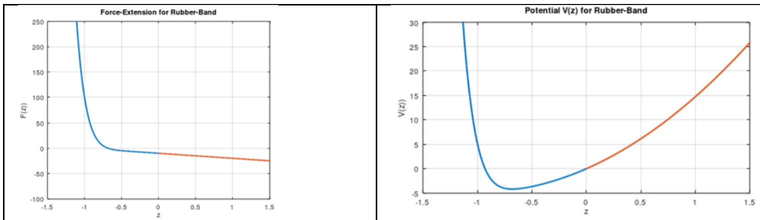



Figure 54. Left shows non-linear force-extension model, right the equivalent potential.

mass-rubber band oscillator in these graphs; this is the value of $z$ where the force is zero and the potential has a minimum and it turns out to be around -0.684m for this example. Now let's consider oscillations around this point; we know that they will be close to harmonic. Fig.55 shows the position and velocity traces; we see sine wave traces and a circular orbit; we have harmonic motion!

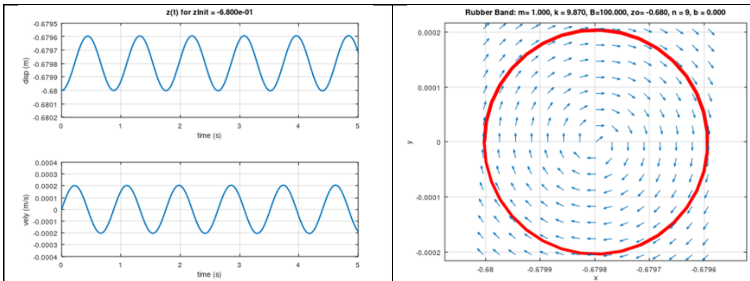



Figure 55. Left position and time traces, right trajectory on the phase plane. These look like they come from a linear oscillator.

Now let's look at a trajectory away from the $z = 0.5$ and we see completely different behaviour. The phase plane is not circular, which tells us we do not have harmonic motion. This is confirmed by the $z(t)$ graph, which is clearly non-sinusoidal, Fig.56. Let's think about this.

This plot tells us a lot. The tops look rather sinusoidal, but the bottoms are narrowed, peaked. This reflects the non-linearity at work; near the top we have compression, and the force is linear, near the bottom we have extension, and the force is non-linear.

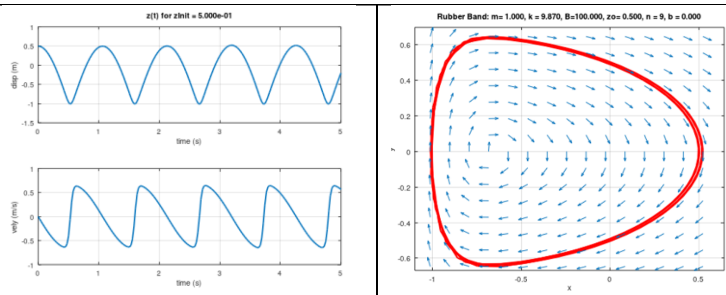



Figure 56. Time traces and trajectory in a non-linear region.

The phase plane also betrays this distinction; at the right we have a more 'circular' curve while at the left it is squished with a larger vertical component.

Now let's estimate the frequency of small-amplitude oscillations near the equilibrium point, using eq.54 and with $dF(z)/dz = -k - 9\alpha z_{Equ}^{8}$ using the value $z_{Equ} = -0.684$ we find this gives us a period of $T$ = 0.86 sec.

We can numerically compute the periods of large-amplitude oscillations using the 'energy method', shown in section?? Here is a table of numerical results

| initZ | T |
|---|---|
| 1.5 | 1.07 |
| 1.0 | 1.07 |
| 0.5 | 1.06 |
| 0.0 | 1.04 |
| -0.68 | 0.88 |

There are a number of interesting things here. First, we see that the value for -0.68 has a period which agrees with our approximation for small-amplitude oscillations, 0.86 sec. Then for large initial displacements, the period is pretty constant at 2.07 sec, and is approximately half of the period of a quadratic potential, $T = 2\pi\sqrt{m/k}$ = 2.00 sec. So we have support for our assertion in section 1.5.4.

There is also another important point here concerning the *amplitude* of oscillations, as defined by half the value of maximum and minimum displacements. This actually has little relevance to such a non-linear oscillator, especially when we are using the energy approach to estimate the periods. The values in Table.1 used the initial displacement as the independent variable to characterize an oscillation trajectory. This has much more physical sense here than the formal amplitude since it is directly related to the system energy.

### 1.11.2 A Piezoelectric Energy Harvester

As we potentially move towards the ‘Internet of Things’ consisting of small remote wireless-connected devices such as light switches, thermometers and intrusion sensors, a problem emerges, how to power them? A typical home will probably contain a hundred or so devices, so to power them by batteries would become inconvenient. We need an alternative way of localized power. Localized power is in itself interesting since it may lead to electrical installation savings. If all of our light switches communicated by radio signals with a central controller, then there would be no need for hundreds of metres of copper wire connected to those switches.

One interesting device currently in research and development uses a *piezoelectric* strip, a thin semiconductor sandwich which produces several volts of output when the end moves by a few mm and the strip bends. As shown in Fig.57 one end of the strip is fixed to some structure, and the other end supports a mass which is free to move.

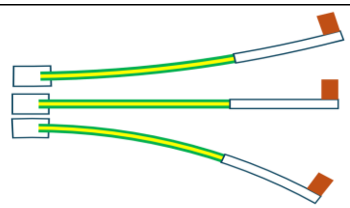


Figure 57. Snapshots of oscillations of a piezoelectric energy harvester.

The strip has a certain stiffness, so our system is like a mass on a spring with one end of the spring forced. Yep, this is forced harmonic motion! The system will have a natural frequency of oscillation, so when the excitation frequency is close to this, we expect resonance; the mass will have a large oscillation amplitude. The strip will then have a large amount of bending and produce a large voltage.

A typical research laboratory setup is shown in Fig.58 which we have simulated in PhysLab. An electromagnetic vibrator drives one end of the strip; some electronics ensures that the acceleration of the end always has an amplitude 1g. The mass is free to oscillate, and the voltage generated is measured.

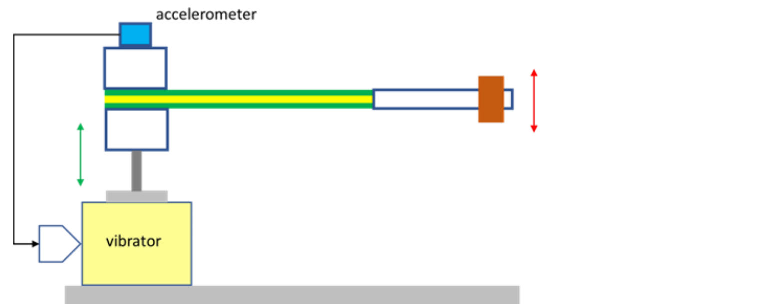



Figure 58. Research setup to investigate a PIE.

How do we set up the equation of motion for this system which will tell us how the output voltage (and therefore delivered power) varies with time? We must start with the *cause* of the voltage which comes from the bending of the strip, and the simplest model of this bending is a spring, where the restoring force is proportional to the displacement of the mass. We must also include damping, and we have stated that the amplitude of the excitation is 1g. This leads to the equation of motion

$$\ddot{z} = -\frac{k}{m} z - \frac{b}{m}\dot{z} + g\cos\omega t. \qquad (165)$$

From eq.129 we know that the amplitude of the steady-state response to the forcing is just

$$A(\omega) = \frac{g}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \qquad (166)$$

where $\beta = b/2m$. This is a straightforward application of our theory. Let's discuss an actual example of a possible research project to design an energy harvester.

**Example ??** The frequencies of ambient vibrations such as household equipment, cars, buildings and humans are usually less than 100Hz. We wish to design a system with a natural frequency in this range. Given the constraints of fabricating a realistic device, we choose a mass $m = 9.62 \times 10^{-4}$kg and stiffness $k = 25.8$ N/m. Experiments give a damping coefficient $b = 6.3\times 10^{-3}$ kg/s. With these values, you can confirm that the resonant frequency is $f = (1/2\pi)\sqrt{k/m}$ which is 26Hz. So far so good.

From eq.?? we find the amplitude of oscillation when we excite at resonance is ??. For our device, experiment shows that the generated voltage is related to amplitude by a factor 156.25, so the voltage produced is ?? and if connected to a 50kOhm resistor produces a power ?? While this may seem low, it is enough to power a small sensor and transmitter.

But we also have to consider the frequency response curve of this device, which we can plot from eq?? We can see this is quite sharp [something about bandwidth here]. So this is a limitation; our device is so accurately tuned to ??Hz, that it will not respond to (and harvest) other frequencies present in the ambient vibration range of up to 100Hz. We shall address this in the next section.

### 1.11.3 A Multi-Piezoelectric Energy Harvester

We know that ambient vibrations have frequencies up to 100Hz, but our (motivated) design presented in Example?? had a relatively small *bandwidth* and was therefore able to extract just a small proportion of the power available. How can we improve on this? One obvious approach (which is being researched) is to design a system with multiple strips, with different resonant frequencies which together cover the range up to 100Hz.

We can easily do this by changing the mass attached to each strip, see Fig.59 So our exemplar device is shown at the bottom with a resonant frequency of ??. Moving upwards, the mass is decreased so the resonant frequency increases. You can see how, with careful choice of masses, we can design a system which might be able to get from ??Hz to 100Hz.

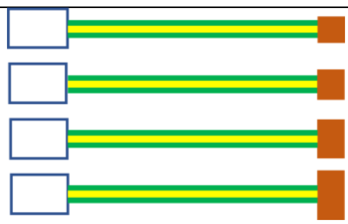


Figure 5. Multi-PIE VEH with range of resonant frequencies; highest (top) to lowest (bottom).

Let's look at the results of such a system design and then explore how we have designed it. Figure.60 shows the frequency response of a four-strip device where the amplitudes of each strip are plotted; vertical dashed lines show the designed resonant frequencies.

Our single PIE device is shown by the blue (leftmost) curve. Let's compare the single and multi-PIE devices by looking at the frequencies which yield an amplitude of 4mm. The single device has a response range of around 25-27 Hz and the multi device has a response range of around 25 – 32Hz. So there is a dramatic improvement, so we have an *engineering* success!

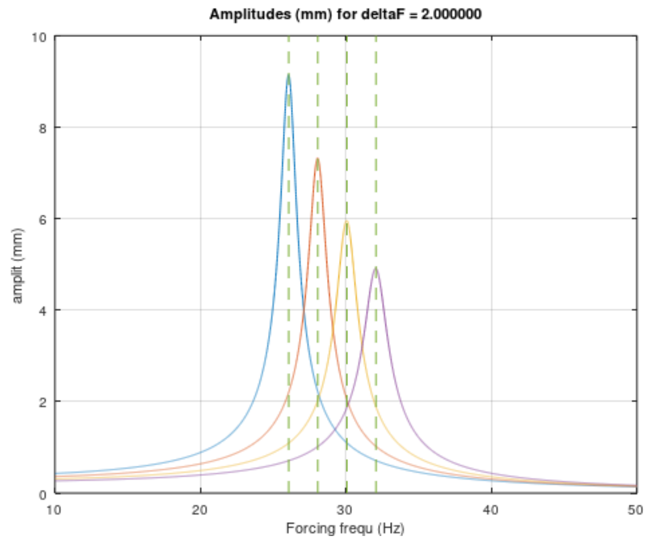



Figure 60. Response of a multi-PIE VEH showing individual response curves.

Now let us think about this physics expressed in Fig.60. Suppose we have our four devices, and we excited them together starting with a frequency of 20 Hz and steadily increased this up to say 40Hz. What would we see if we *directly observed* the apparatus? Well, we follow the plots. At 20Hz most elements do not vibrate a lot, but when we hit 26Hz the first element is really vibrating. Moving upwards in frequency this element vibrates less, then the second starts to increase and dominates the system movement at 28Hz. Moving upwards the second element starts to decrease and the third takes on the baton and dominates at 30Hz. You get the idea; each element has its own resonant frequency and so extracts energy from the ambient energy where it can.

In terms of the mathematics and ODE formulation, there is little to add here; each element will obey its own eq.165 and eq.166.

### 1.11.4 The 'Gravity Tube'

This is a hypothetical experiment where a mass is dropped into a 'gravity tube' which has two halves; in the top half gravity is pointing down (as usual) but in the bottom half gravity is pointing upwards. This is sketched in Fig.61 where a possible approximate piece of physical apparatus is shown.

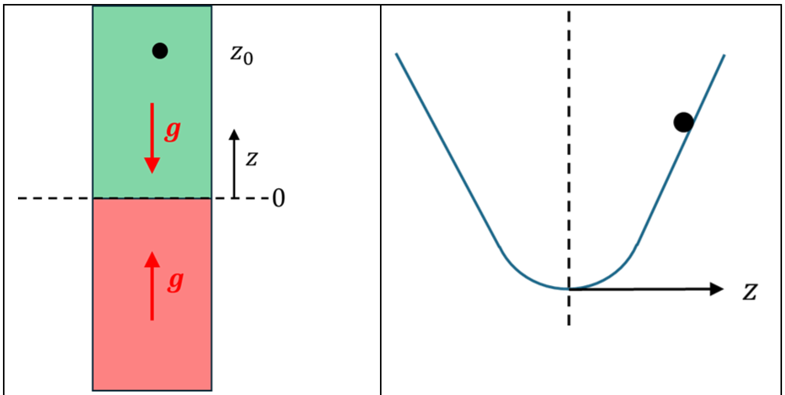



Figure 61. Left shows the hypothetical tube as found in PhysLab, right shows a close implementation as a ball on a track.

The magnitude of both gravity vectors is identical, and we assume that there is no friction. There is clearly a *discontinuity* where the tubes meet. The apparatus consists of a ball rolling along a track which rises linearly with displacement on both sides of $z = 0$; the tracks are joined smoothly by a short, curved segment to remove any discontinuous join.

To understand the dynamics of this system, we do not look for an equation of motion expressed as an ODE due to the discontinuity; in fact, we don't have to, since we can apply the equations of kinematics directly. To find the time $t$ taken for the mass to fall from its initial displacement to the boundary at $z = 0$ we have

$$z_0 = \tfrac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2z_0}{g}}. \qquad (167)$$

By symmetry, the time taken in the lower half from the boundary to the rest position is the same, therefore the period of oscillation (from starting point back up to starting point) is just

$$T = 4\sqrt{\frac{2z_0}{g}}\,. \qquad (168)$$

For a typical experiment with a starting displacement of 2m, with the usual value of gravity, we find a period of 2.55 seconds.

The importance of studying the gravity tube is that it provides us with an oscillator which is **not** harmonic; as you can see from eq.?? the period depends on the ‘amplitude’ of oscillations, here the starting value $z_0$. If we solve eq.168 as a function of time for the position of the mass, in the top half, we have

$$z(t) = z_0 - \frac{1}{2} g t^2, \qquad (169)$$

which describes a parabola. So we expect the oscillation time-trace to be a series of connected parabola, see Fig.62. From eq.169 we have an expression for the velocity

$$v(t) = -gt, \qquad v(0) = 0, \qquad (xx)$$

where the velocity time-trace is a series of straight lines. Eqs.169 and 170 are clearly very different from the equations of simple harmonic motion which have the form $z(t) = A\sin(2\pi t/T)$ and $v(t) = -(2\pi/T)\cos(2\pi t/T)$.

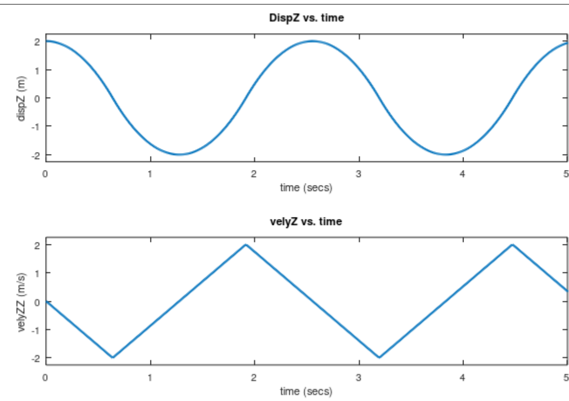



Figure 62. Time traces for the Gravity Tube oscillator.

The first (negative) velocity peak occurs at time $t = \sqrt{2z_0/g}$. We have rescaled velocity $v' = v/\sqrt{g}$, to make the phase plane more ‘circular’, so the value of this first velocity peak is

$$v'_{peak} = \frac{-gt}{\sqrt{g}} = -\sqrt{g}\sqrt{\frac{2z_0}{g}} = -\sqrt{2z_0}. \qquad (171)$$

We note discontinuities in the velocity trace which are harder to spot in the displacement trace, but they are very visible in the phase plane, Fig.63 This shows smooth changes in $z$ and $v$ except at $z = 0$ where the displacement changes in sign. Changes in the sign of the velocity occur at displacement extrema (top and bottom of the combined tube), and these are continuous.

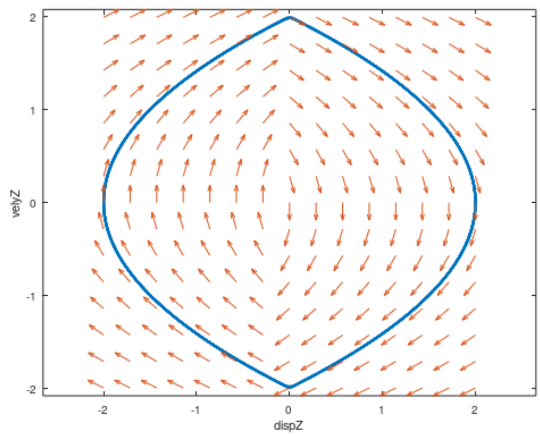



Figure 63. Phase-plane trajectory for the Gravity Tube.

In the above analysis we have taken the initial conditions as (i) an initial displacement, (ii) zero velocity. Two ICs are required since this is a 2nd-order dynamical system. Here we look at the complementary ICs where we have zero initial displacement but an initial velocity $v_{init}$, see Fig.64

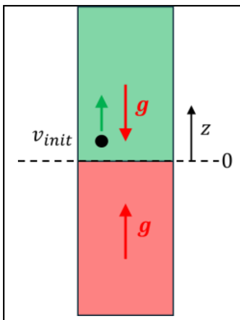



Figure 64.

To understand this situation, we use the initial velocity to calculate the height of the mass when it has stopped, then we have the previous situation. Using energies

$$\tfrac{1}{2} m v_{init}^2 = mgz \quad \Rightarrow \quad z = \frac{v_{init}^2}{2g}, \qquad (xx)$$

which, using eq.168 gives us

$$T = 4\frac{v_{init}}{g}, \qquad (xx)$$

so we find the period is proportional to the initial velocity. This is a nice clean result.