Chapter X
Double Well Oscillator

X.1 A Brief Introduction

Here we are going to look at a somewhat strange, almost hypothetical system where the linear component of the restoring force is positive, i.e., it is not restoring. The nonlinear term is pure cubic which allows analytical solutions in many cases, e.g., for the bifurcation diagram, and for small-amplitude oscillations. It is based on the *Duffing equation*, well studied and reported in the literature. Moreover, it is an *archetypal* system, since all systems where the nonlinearity is odd can be very accurately approximated by the double-well through a process called *cubification*, though this is advanced material, beyond our current needs.

We can visualize the system as a mass tethered between two strange, hypothetical springs, as shown in Fig.1, where we also depict the potential curve where a small particle is shown traversing the curve.

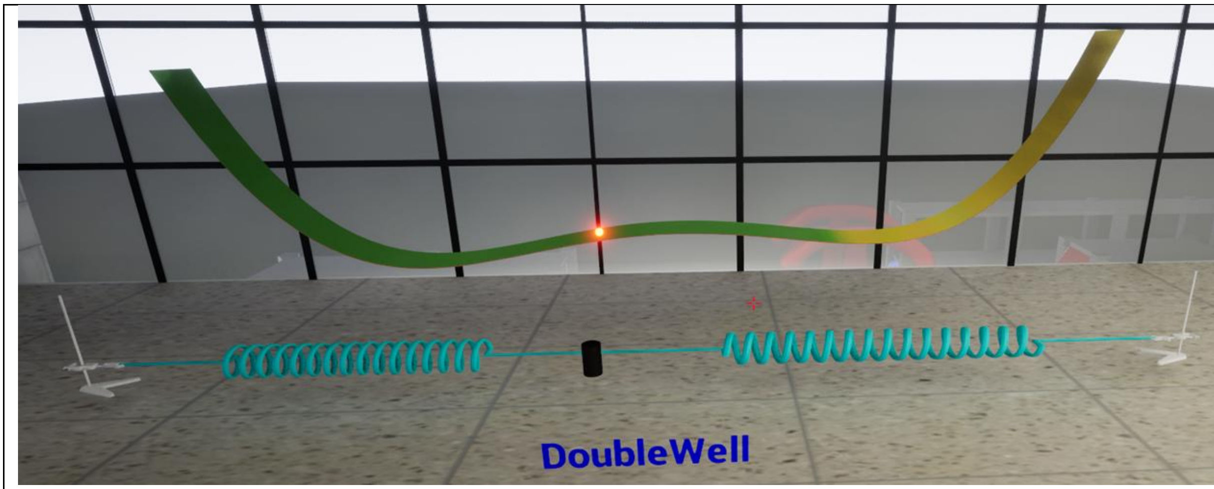

Figure 1. Mass-spring oscillator with potential curve showing two equilibrium positions.

X.2 Equations of Motion

X.2.1 Forces and Fixed points

We start with the stated equation of motion, not related to any concrete physical system,

$$m\ddot{x} = kx - x^3 \quad (1)$$

where the force (right-hand-side) is sketched in Fig.2. The fixed points of the system (force is zero) are shown as yellow dots, and the gradient of the force is indicated. Clearly we have an unstable equilibrium at $x = 0$ and stable equilibria at positions

$$x_{equ} = \pm\sqrt{k} \qquad (2)$$

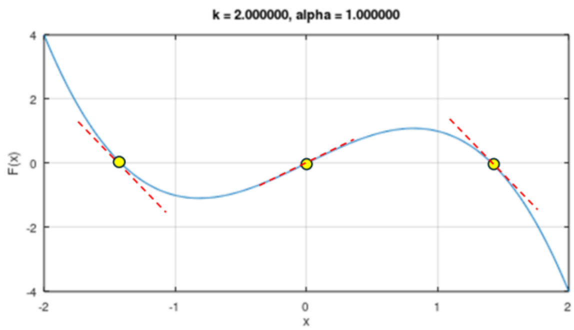

Figure 2. Force extension curve for our system showing a central unstable equilibrium (as deduced from the gradients shown) and two stable equilibria.

X.2.2 Frequencies of oscillations.

It is straightforward to obtain the frequencies of oscillation around the fixed points, at least for small-amplitude oscillations. We can proceed in two ways. First we can use the idea that the frequency of harmonic oscillations is given by

$$\omega = \sqrt{\frac{k'}{m}} \qquad (3)$$

where k' is the effective stiffness of the spring near the fixed point. This is of course given by

$$k' = -\frac{dF(x)}{dx} \qquad (4)$$

which for $F(x) = kx - x^3$ with $x_{equ} = \sqrt{k}$ results in

$$\omega = \sqrt{\frac{2k}{m}} \qquad (5)$$

which corresponds to the hypothetical-physical arrangement depicted above.

A second approach involves expanding the solution $x(t)$ around the fixed point x_{equ}

$$x(t) = x_{equ} + \varepsilon(t) \qquad (6)$$

where we assume the second term is relatively small. Substituting into eq.(1) we have

$$m\ddot{\varepsilon} = k\varepsilon + kx_{equ} - \left(x_{equ} + \varepsilon\right)^3 \quad (7)$$

Expanding, and keeping only the lowest order term in ε we end up with

$$m\ddot{\varepsilon} = -2k\varepsilon \quad (8)$$

from which eq.(5) follows.

X.2.3 The Bifurcation Equation

Here we are interested in investigating how the equilibrium solutions $\pm x_{equ}$ and the frequencies of small-amplitude oscillations around these, vary with a principal system parameter which plays the role of the bifurcation parameter. The available parameter is of course k. The required equation is eq.(2) which is shown in Fig.3. Those familiar with bifurcation theory will recognize this as a super-critical pitchfork bifurcation, with a critical point $k = 0$.

The frequencies above the critical point are given by eq.(5). Below the critical point the dynamics become

$$m\ddot{x} = \left. -kx - x^3 \right|_{x=0} \quad (9)$$

and using eq.(4) about $x = 0$ we find the frequencies below the critical point

$$\omega = \sqrt{\frac{k}{m}} \qquad (10)$$

Plots of the equilibrium values and frequencies as a function of k over a range of k from 0 to 2 are shown in Fig.4 together with experimental results.

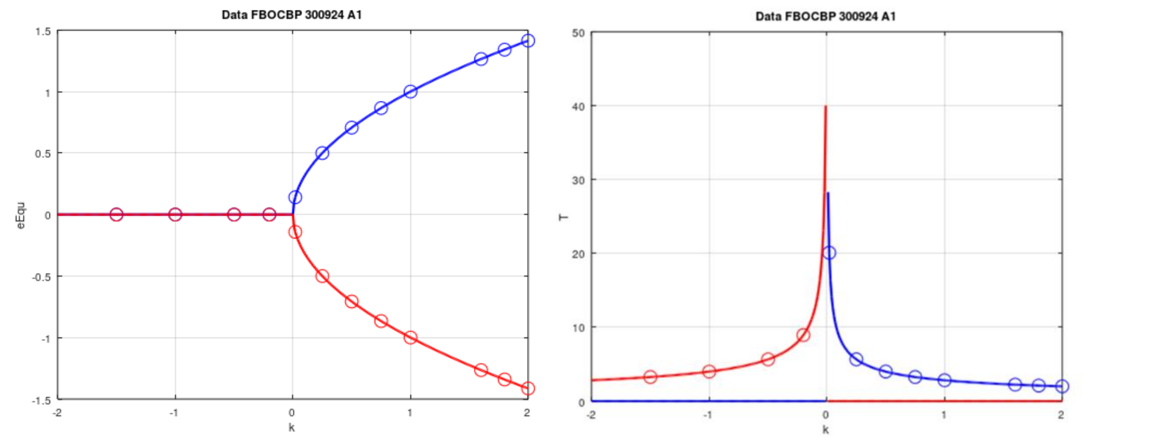

Figure 3. Theoretical equilibrium positions (left) and periods (right) as a function of k together with some experimental results (circles).

X.5 The Potential Hill

X.5.1 A short amble around a Potential Landscape

So far, we have been considering behaviour near the equilibrium points, and have focussed on small-amplitude oscillations. Working with the potential hill helps us to understand large amplitude oscillations. We think of our mass as moving in a field described by the potential $V(x)$ which is related to the force on the mass,

$$F(x) = -\frac{dV(x)}{dx} \qquad (11)$$

Before we set about deriving the expression for $V(x)$ let's take a look at its appearance, Fig.4, where we have related it to the force expression. The red dashed lines indicate the fixed point where the force is zero, these correspond to the lowest points in the potential, at the bottom of its wells. A particle placed here would stay put, since it cannot autonomously roll uphill.

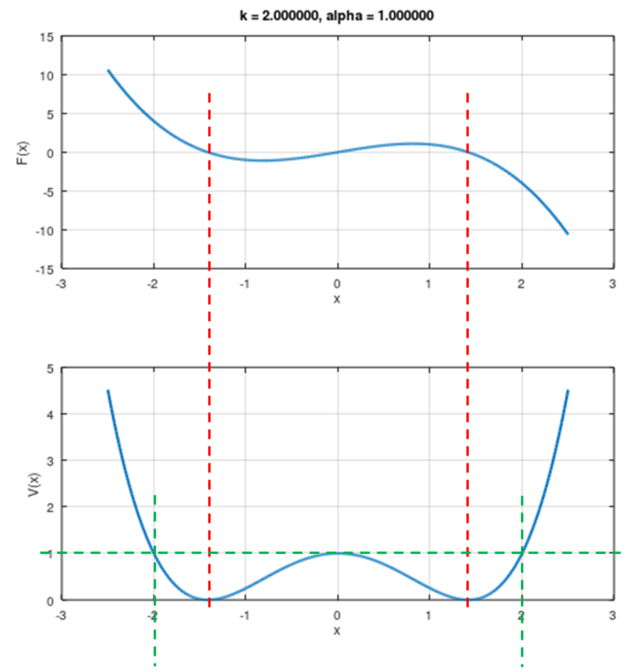

Figure 4. Typical potential hill (bottom) with corresponding force-extension curve (top) for the parameters indicated.

But the potential plot also gives us some additional information. If a particle started at x = -2, then it would roll down the left well and back up the other side until it reaches its starting height. Then it would be at x = 0, and would roll back down the left well, and so continue oscillating. So a particle of initial potential = 1 would just remain bounded in the left well, and any particle of lower potential would remain here too. While the trajectories of these bound particles will in general not be harmonic, we expect many to be quite close.

On the other hand particles with initial potential above 1.0 would traverse both wells, and their trajectories will certainly not be harmonic.

Let’s have a look at some bound solutions, Fig.5, for three initial starting potentials (and therefore three different starting displacements).

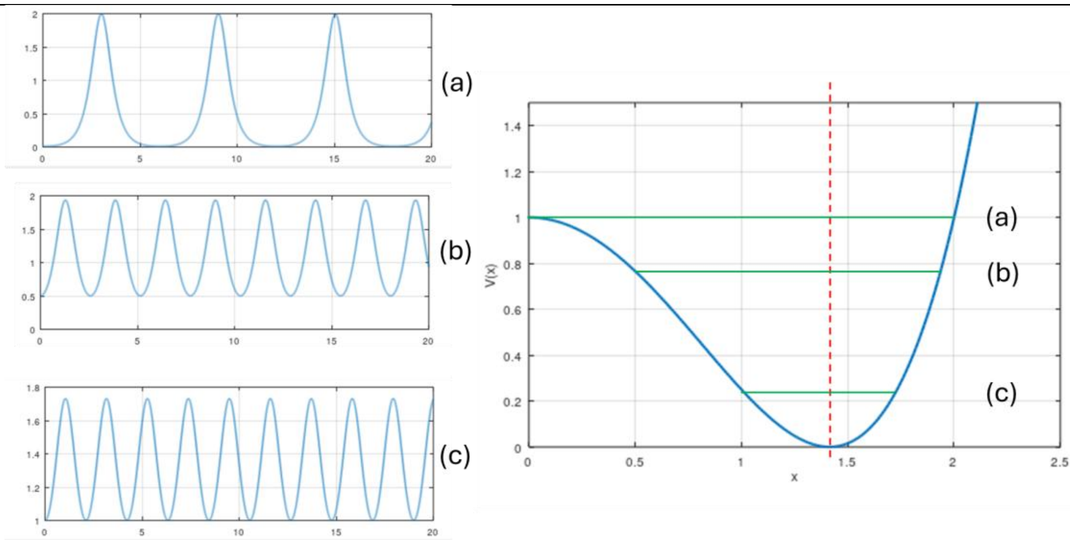

Figure 5. System solutions for the initial conditions shown on the potential hill.

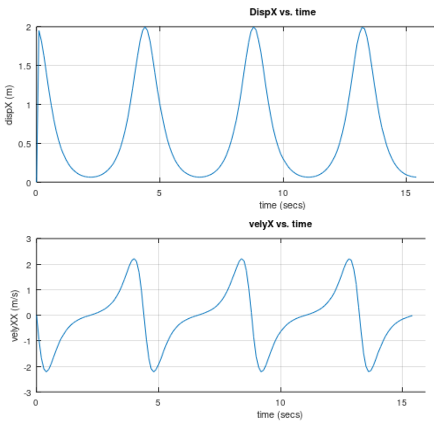

Figure 6 Trajectory close to potential hump.

Progressing downwards we see two different changes in the trajectories. First, they are increasing in frequency and second, they are changing in shape. Let's first consider frequency. The period of the bottom plot close to the bottom of the potential well is around 2 seconds. The corresponding frequency (for $k = 2$N/m, $m = 0.4053$ kg) agrees with eq. (5). More on this later. As we move up the series the period increases. We can sort of understand this, since the velocity at the bottom of the well will be proportional $\sqrt{V_{start}}$ where V_{start} is the particle starting potential. The distance travelled increases faster than linear, so we expect the period to increase.

It's also straightforward to get a qualitative understanding of the trajectory shape change. Near the bottom of the well the particle spends about the same time on either side of the equilibrium, its trajectory is near symmetric and probably harmonic. Near the top of the well the potential is more asymmetric, the particle spends more time on the left which is reflected in the larger time with small x-values, leading to the "cnoidal" shape.

These ideas are confirmed by the velocity-time plot for the oscillator, Fig.6 shows an example for an initial displacement of 1.99m which puts the oscillator very close to the energy of the potential hump. You can clearly see regions where the velocity is small, this lengthens the period and distorts the shape.

X.5.2 The Form of the Potential

Now it's time to derive the form of our potential using eq.(11). Performing the integration we have

$$V(x) = -\int_0^x (kx - x^3)\,dx \qquad (12)$$

$$= -\frac{1}{2}kx^2 + \frac{1}{4}x^4 + cst \qquad (13)$$

where we are free to choose the constant of integration. We choose this to be $\frac{1}{4}k^2$ so the zero of potential is located at the bottom of the wells. This leads to a simple expression for our potential which has been used in the above plots

$$V(x) = \frac{1}{4}(x^2 - k)^2 \qquad (14)$$

It is straightforward to deduce that the well bottoms are located at $\pm\sqrt{k}$ and the height of the potential hill is of course $\frac{1}{4}k^2$. A typical plot for $k = 2$ is shown in Fig.7

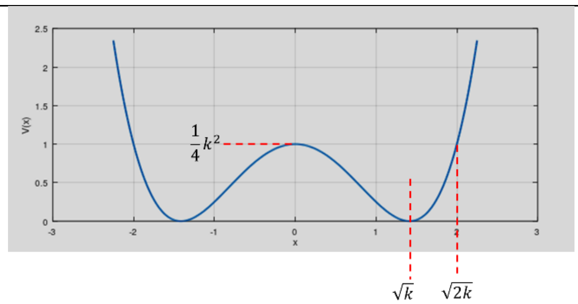

Figure 7. Double Potential Well.

While we are busy with the maths, let's try to model the potential around one well, in particular, let's fit a *quadratic* curve to the bottom of the well. That's useful since it gives us another way to calculate the oscillation frequency near the equilibrium points. The idea is shown in Fig.8 where we plot the full potential $V(x)$ and an approximating quadratic potential $V^{(2)}(x)$ around the equilibrium $x = \sqrt{k}$.

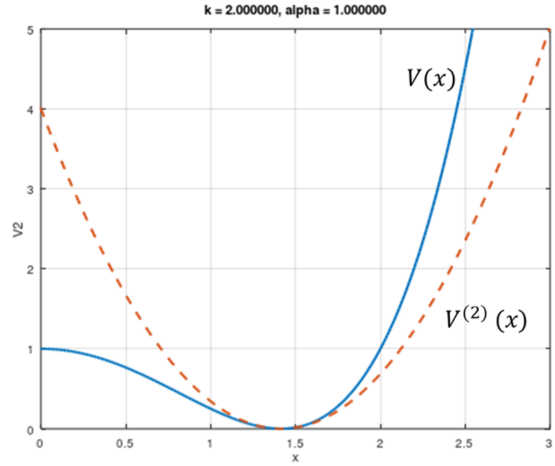

Figure 8. Full potential $V(x)$ and quadratic approximation $V^{(2)}(x)$ around the rightmost equilibrium point. Dotted curve shows the approximation.

We construct the Taylor expansion

$$V^{(2)}(x) = V(a) + V'(a)(x-a) + \tfrac{1}{2}V''(a)(x-a)^2 + \cdots$$

with $a = \sqrt{k}$. It is easy to show that $V(a) = 0$ and $V'(a) = 0$, so we are left with the second derivative term. It follows that

$$V^{(2)}(x) = k\left(x - \sqrt{k}\right)^2 \qquad (15)$$

which is the red dashed curve plotted above. Now we can use eq.(4) to find the oscillation frequency

$$F(x) = -\frac{dV^{(2)}(x)}{dx} = 2k\left(x - \sqrt{k}\right) \qquad (16a)$$

$$k' = -\frac{dF(x)}{dx} = 2k \qquad (16b)$$

which agrees with our previous result.

X.6 Large Amplitude Solutions

For a nonlinear system it is the large amplitude solutions which are of particular interest. Unlike linear systems, where the amplitude of oscillation is defined by the initial conditions, nonlinear systems behave completely differently. Here the large amplitude solutions are defined by the system itself, in other words the system parameters.

There are several available mathematical approaches, here we apply a quite recent approach, the *energy balance* method.

X.6.2 Global Solutions

Here we consider solutions where the oscillations traverse both potential wells shown in Fig.9.

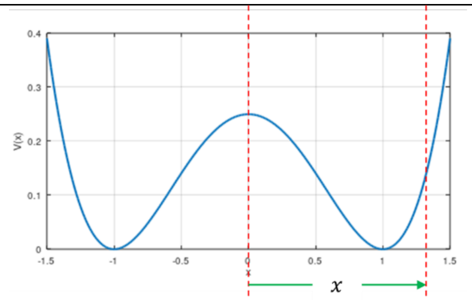

Figure 9. [needs revising]

Starting from the equation of motion (slightly re-ordered),

$$m\ddot{x} - kx + x^3 \qquad (17)$$

we can obtain an expression for the total energy of the system

$$\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 + \tfrac{1}{4}x^4 = cst \qquad (18)$$

now take two points on the trajectory, first at its beginning where we have $x(0) = A$ and the second at a general point. The total energy at these points is the same, so we have

$$\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 + \tfrac{1}{4}x^4 = -\tfrac{1}{2}kA^2 + \tfrac{1}{4}A^4 \qquad (19)$$

Now we take a trial function

$$x = A\cos\omega t \qquad (20)$$

and substituting into eq.(19) we find after some cleanup

$$m\omega^2 \sin^2\omega t - k\cos^2\omega t + \tfrac{1}{2}A^2\cos^4\omega t = -k + \tfrac{1}{2}A^2 \qquad (21)$$

Now we choose a point on the trajectory and assert that the energies are equal at this point. They cannot be equal at *all* points along the trajectory unless our trial function eq.(20) is the correct solution. To avoid losing any terms in eq.(21) we choose $\omega t = \pi/4$ which is 1/8$^{\text{th}}$ the way along the trajectory. With some cleaning eq.(21) becomes

$$m\omega^2 = \frac{3}{4}A^2 - k \qquad (22)$$

and so we find the relationship between frequency and amplitude of the oscillation

$$\omega = \frac{1}{2}\sqrt{\frac{3A^2}{m} - \frac{4k}{m}} \qquad (23)$$

There is of course a limitation, we must have $A^2 > (4/3)k$, so the amplitude must be sufficiently large!

Despite the simplicity of this approach, the results are rather impressive, Fig.10 shows experimental results for *m=0.4053* kg and *k=1* N/m. The blue line follows the above analysis, the red line is experimental, and follows our current research into how to optimally choose the co-location angle.

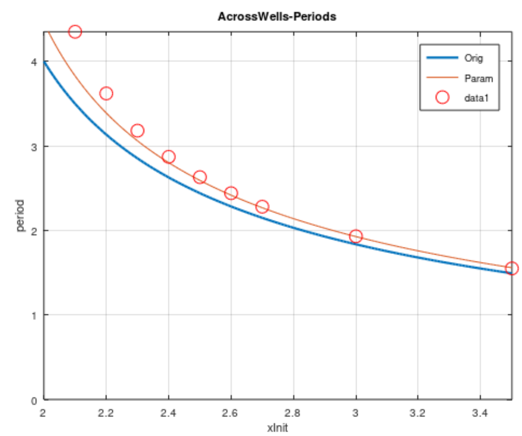

Figure 10. Period versus initial displacement according to the Energy Balance method (blue line) with some simulation data (circles).

A small catalogue of solutions is reproduced in Fig.11.

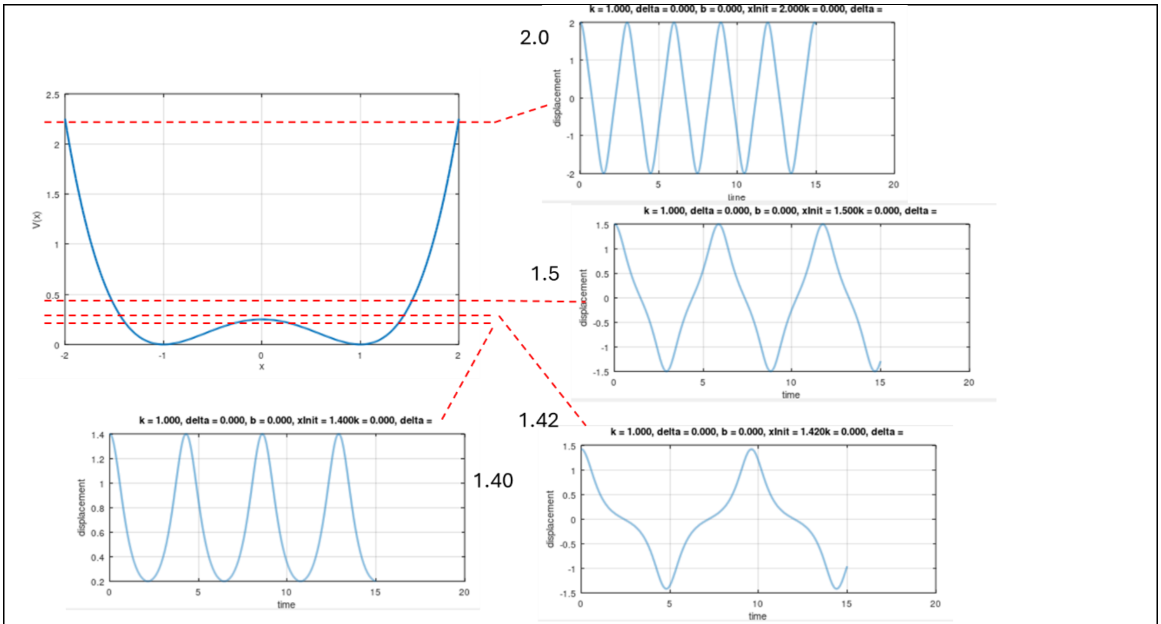

Figure 11. Catalogue of solutions for various potentials shown.

Starting with the amplitude 2 solution, the inter-valley rise is very small, so the mass starting from the left will pick up speed which will remain more or less constant for most of its cycle. You can see this from the x-t graph which comprises linear segments (constant velocity) between rounded ends. As the amplitude is reduced, the effect of the rise becomes more dominant and there are significant times when the velocity is small. Finally the solution with amplitude 1.40 remains confined to a valley.

X.6.3 Local Solutions

Here we are interested in large amplitude solutions which are confined to a potential well described by variable y as shown in Fig.12 for $k = 2$.

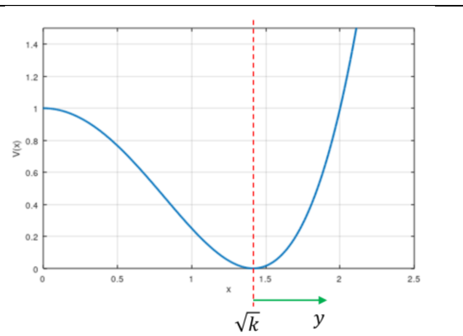

Figure 12. Diagram to aid in establishing large-amplitude solutions within a potential well

Our task is to establish a suitable potential function $V(y)$. There are two constraints on the associated force: First, to agree with the small amplitude approximation, the gradient of the force around the equilibrium location $y = 0$ ($x = \sqrt{k}$) must be $-2k$. Second, the force at $y = -\sqrt{k}$ must be 0. This leads to the following force and potential

$$F(y) = -2ky + 2y^3 \qquad (24)$$

$$V(y) = ky^2 - \tfrac{1}{2}y^4 \qquad (25)$$

Applying the above energy balance method results in the following frequency-amplitude relationship

$$\omega = \sqrt{\frac{2}{m}}\sqrt{\frac{4k}{m} - \frac{A^2}{m}} \qquad (26)$$

Again the results are rather impressive, Fig.13 shows experimental results for *m=0.4053* kg and *k=1* N/m.

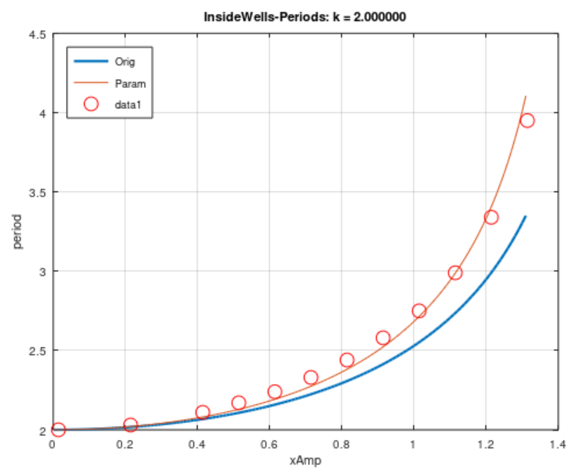

Figure 13. Large amplitude solutions within a potential well by the Energy Balance method (blue line) with some simulation results (circles).

It's interesting to compare Figs. 10 and 13 which show the amplitude having opposite effects on the period. Within a well period increases with amplitude, as we have already said, this is due to the average oscillator velocity reducing with amplitude. When the oscillation is across both wells, the period reduces with amplitude. We suggest this is due to the hump having smaller influence over large amplitude cycles.

X.7 Unfolding the Bifurcation curve

Many physical systems can be modelled by the introduction of a constant offset to the force, the equation of motion now reads

$$m\ddot{x} = kx - x^3 + \delta \quad (27)$$

The equilibrium solutions are obtained from

$$kx - x^3 + \delta = 0 \quad (28)$$

for which exact analytical solutions are available. You may think that periods of small-amplitude oscillations are the same without the offset, since they depend on the derivative of the expression for the force. But this is incorrect, since the periods must be evaluated around the equilibrium positions of the oscillator, and these depend on δ. Fig 14 illustrates this for two values of δ where *k = 2*. The potential is now

$$V(x) = \frac{1}{4}(x^2 - k)^2 - \delta x \qquad (29)$$

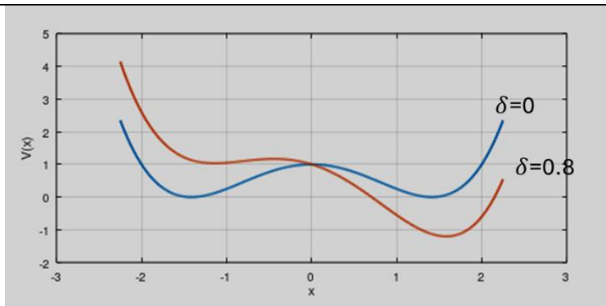

Figure 14. Potential hill with and without unfolding.

It's straightforward to derive an expression for the frequencies as a function of x_{Equ}

$$\omega^2 = -\frac{\left(k - 3x_{Equ}\right)}{m} \qquad (30)$$

where x_{Equ} can be obtained by solving eq.28

The unfolded bifurcation curve obtained from eq.(28) together for some experimental solutions for k = 2 N/m and $\delta = 0.1N$ are shown in Fig.14.

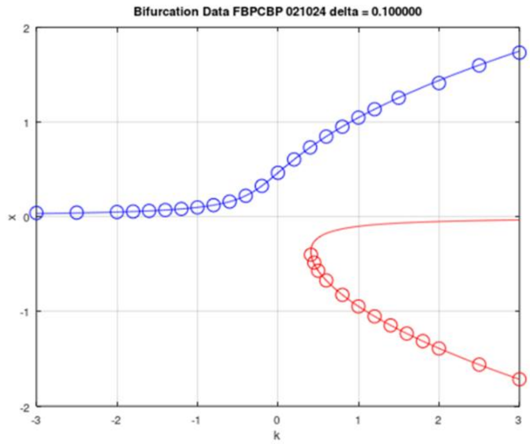

Figure 14. Unfolded bifurcation curve due to parameter $\delta = 0.1$

Comparing this with Fig.3 we see the lower branch has become ‘detached’ from the upper branch and there is a range of equilibrium values which the system cannot produce.

A typical investigation of the lower branch would start by obtaining a solution for $k = 3$, then slowly reducing k to trace out the branch. You would find that the equilibrium value would continually reduce until the point $k = \sqrt[3]{27\delta^2/4}$ at which point the solution would *jump* to the upper branch. This is detailed in Chapter.XX. The critical value of k is obtained from analysis of the cubic equation. Results of an investigation of this jump are shown in Fig.15

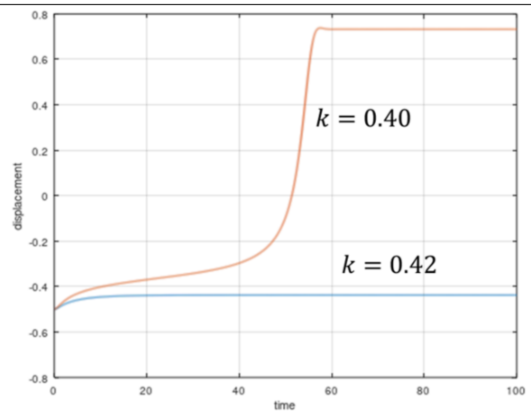

Figure 15. Demonstration of the jump phenomena near the critical value of k. Both solutions start with the same initial conditions, -0.5. As time evolves, one solution remains on the lower branch while the second jumps to the upper branch.

Two experiments were performed for identical initial conditions x = -0.5 and bracket the critical k. The jump for k = 0.4 is clearly visible.

We can also present theoretical curves and experimental results in additional ways, which may lead to insights. Fig.16 shows the period of small-amplitude oscillations as the bifurcation parameter is varied. The peak in the upper branch is a consequence of the different behaviour above and below k = 0, below we have (for the unfolded bifurcation) $\omega^2 = k/m$ while above we have $\omega^2 = 2k/m$.

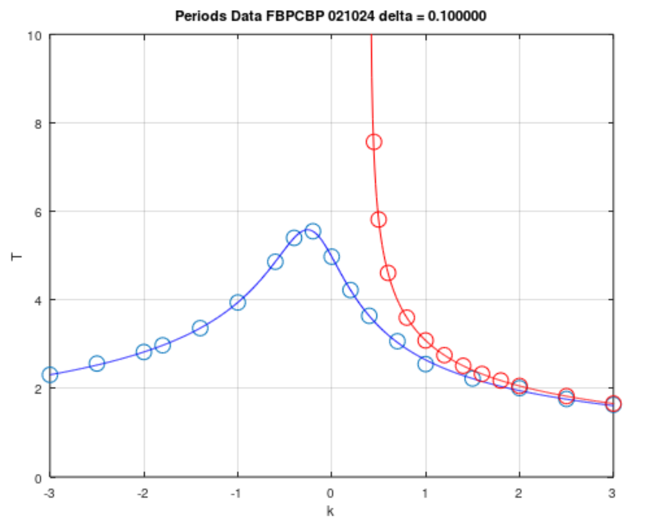

Figure 16. Theoretical and simulation results for period as a function of *k* for both upper curve (blue) and lower curve (red).

The lower branch shows interesting behaviour, the period rises to infinity as k approaches its critical value [needs explaining]. So the oscillator, when on the lower branch is able to show a large range of periods. This may be useful to engineers who use such an oscillator to harvest energy from environmental vibrations, since often the range of periods is imprecisely known, or may show a large bandwidth. The lower branch could be a useful place to explore large-bandwidth harvesters.

Finally we consider a plot of period against equilibrium position, Fig.17. Again, taking an engineer's perspective, we would be interested in devices with fairly small equilibrium positions, since these define the physical size of the device. There are clearly regions of both upper and lower branches where reasonably-sized devices showing a useful bandwidth could be constructed. [Revisit this, perhaps look at power output as f(equilib,T)?].

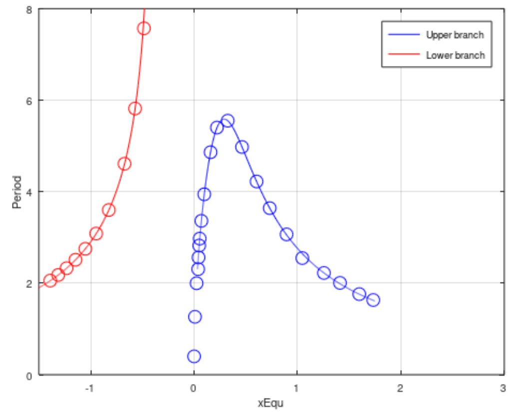

Figure 17. Plots of period against equilibrium positions for the upper and lower branches.

It is interesting to look at a plot of how period depends on δ and we must do this for both wells. First, we must identify the correspondence between wells and branches of the bifurcation diagram, see Fig.18.

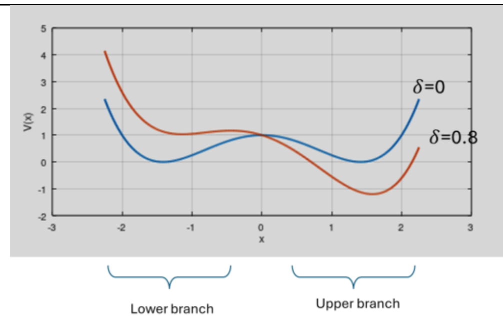

Figure 18

As we proceed along the upper branch, increasing δ, we note that x_{Equ} is increasing, so that the frequency from eq.?? is increasing, hence the period will reduce. On the lower branch x_{Equ} decreases, the frequency will reduce and the period increase. This is visible in Fig.19.

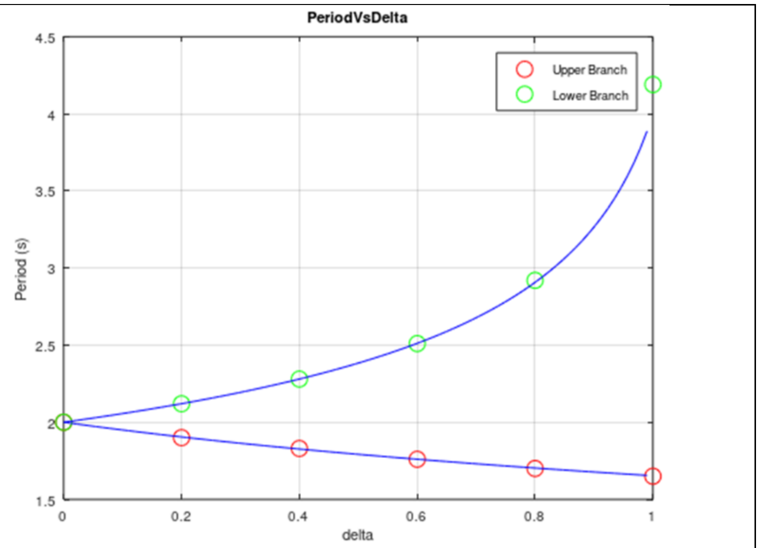
PeriodVsDelta
Upper Branch
Lower Branch
Period (s)
delta
4.5
4
3.5
3
2.5
2
1.5
0
0.2
0.4
0.6
0.8
1

Figure 19.