Chapter X
Fluid Oscillators

X.1 A Brief Introduction

In this chapter we shall take a look at two fluid oscillators. The first is a cylinder floating in water and the second is air in a compressible tube. Both have important engineering applications. An example of a floating (almost) cylinder is the CorPower C4 wave energy convertor which oscillates up and down (heaving motion) in response to ocean waves and contains a mechanism to convert this motion into electricity (Fig.1).

Figure 1

X.2 Floating Bobs

Here we shall study a floating cylinder in a wave tank. The PhysLab simulation arrangement is shown in Fig.2.

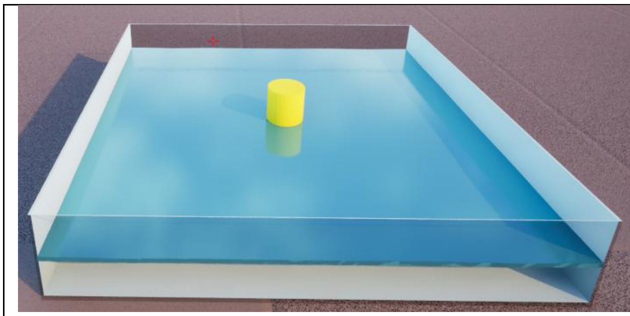

Figure 2 Physlab wave tank simulation.

This is a straightforward system to understand. First, we need to find out how the bob manages to come to a static equilibrium (to float) and then how it behaves when you give it a little displacement. Of course it will oscillate, and so we need to understand the details of this.

X.2.1 Floating and Oscillating

Let’s start with thinking about the static equilibrium. The following discussion may seem a little verbose but it’s important to get all the relevant physics in. Figure 3 shows the forces involved where the cylinder is floating with its submerged draught *d*.

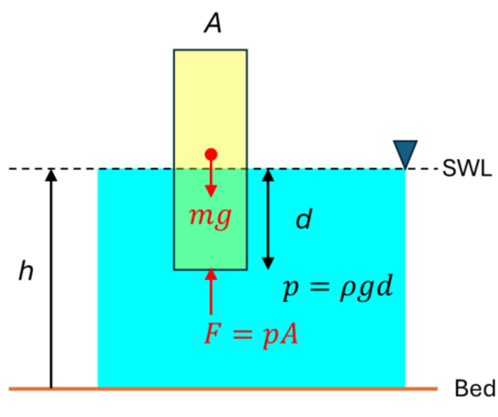

Figure 3.

Let's take our *z*-axis pointing upwards. There is a downward force on the bob $-mg$, its weight. There is also an upwards force which comes from the pressure in the water at depth d. So at equilibrium we have

$$F \;=\; \rho g A d - mg = 0 \qquad (1)$$

From which we can calculate the draught

$$d = \frac{m}{\rho A} \qquad (2)$$

It actually turns out that the draught is a very useful parameter for this system as we shall see. Now let's give our bob a little upward displacement *z* and look at the forces again, Fig.4

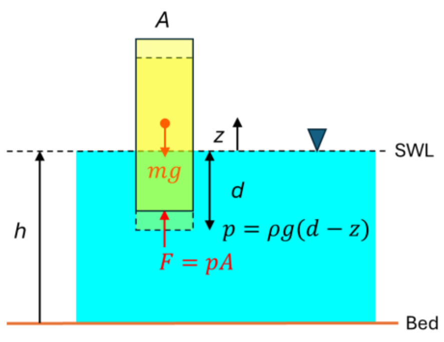

Figure 4

Clearly the pressure is less. The upwards force is now

$$F = \rho g A(d - z) - mg \qquad (3)$$

And using eq.2 we quickly find

$$F(z) = -\rho g A z \qquad (4)$$

Which is a *restoring* force and is linear in z so we expect oscillations! Let's build up the equation of motion. We can immediately write down

$$m\ddot{z} = -\rho g A z \qquad (5)$$

From which we can deduce the frequency of oscillation

$$\omega^2 = \frac{\rho g A}{m} = \frac{g}{d} \qquad (6)$$

Where we have made use of eq.2. This is interesting since it tells us the oscillation frequency depends only on the draught of the bob and not its mass or other dimensions.

While we have made a good start, the frequency calculated from eq.6 does not agree with experiments which report a slightly lower frequency. So we must now develop the theory a little more to understand this, and also to include sources of damping in our equation of motion.

X.2.2 Added Mass

So the frequency calculated from eq.6 is incorrect. Why, because the value of mass m is incorrect. In fact it is too small! The reason is quite simple, when the bob is accelerated it also accelerates some surrounding water. So we must include the *added mass* of the surrounding water into our ODE. So eq.5 must be enhanced as follows where m_a is the added mass due to the surrounding water we must accelerate.

$$(m + m_a)\ddot{z} = -\rho g A z \qquad (7)$$

The question now becomes how to calculate the added mass. To calculate the mass of the surrounding water, we could try to use the mass of the displaced water $\rho A z$, but this does not agree with experiment.

Now things become a little tricky, but much more interesting. We need to draw on advanced theory of hydrodynamics. But first let's have a look at what's going on with the surrounding water as the bob moves, see Fig.5. Here we have indicated two important heights, the 'bed' which is the bottom of our wave tank, and the still water level (SWL). So we are suggesting that the *finite depth of water* in our wave tank is of crucial importance. As the bob executes its heave motion up and down, it excites surrounding water which oscillates, so this surrounding water experiences forced harmonic motion. The bob will alternately push water away then suck it back and this will lead to a *standing wave* in the vicinity of the body.

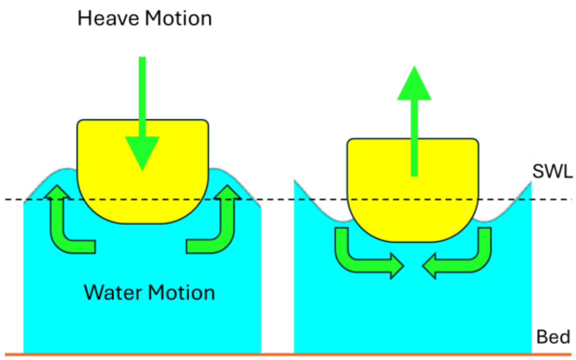

Figure 5. Water acceleration provides added mass.

This will lead to an interaction between the oscillating bob and the standing wave, and there could be *resonance*. At least we can expect that the mass of water accelerated to depend on the bob's oscillation frequency.

What happens if the water depth is reduced while the bob's draft remains the same? Well the bob will be closer to the bed, and there will be much more horizontal acceleration of the water, which suggests the added mass will increase.

It's time to introduce an expression for the added mass derived from advanced hydrodynamic theory[1]. We introduce the following parameters: a is the radius of the cylinder, h is the water depth, d is the draught, λ is the wavelength of the standing waves and γ is Euler's constant, around 0.577. Hold your breath, here it comes. We have colour coded the terms so we can unpick what they mean.

$$m_a = \underbrace{\rho \pi a^3}_{(1)} \left[\underbrace{\frac{a}{8(h-d)}}_{(2)} - \underbrace{\frac{a}{2h}\left[\gamma + \ln \pi \frac{a}{\lambda}\right]}_{(3)} \right] \quad (8)$$

[1] Michael McCormick. Ocean Engineering Mechanics. Cambridge.

We can at once see from term (2) that the added mass dramatically increases as the water depth decreases and approaches the bob's draft. This agrees with our previous thoughts about the water being squeezed through a volume longer but narrower.

The third term is interesting, outside the bracket there is another water depth dependency. But it is inside the bracket that things become really interesting since it depends on the wavelength of the standing waves produced by the bob. So we must understand what determines the wavelength. We suspect this has something to do with the period of our bob.

In our initial discussion we started out looking at the frequency (and therefore the period) of the bob's oscillation, eq.6. So it seems reasonable to take period T as our parameter of interest. For water waves the relationship between period and frequency is

$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{\lambda}\tanh\left(\frac{2\pi}{\lambda}h\right) \qquad (9)$$

You can immediately see a problem solving this equation for λ since it both inside and outside the tanh function. Fortunately for the case of large wavelengths the argument of the tanh is small, and tanh approaches 1. Then we have the 'long wavelength' or 'deep water' limit

$$\lambda = \frac{g}{2\pi}T^2 \qquad (10)$$

To give you confidence in applying this approximation, glance at Fig.6 where the exact and approximate relationships are compared, all looks good over this chosen range of period T.

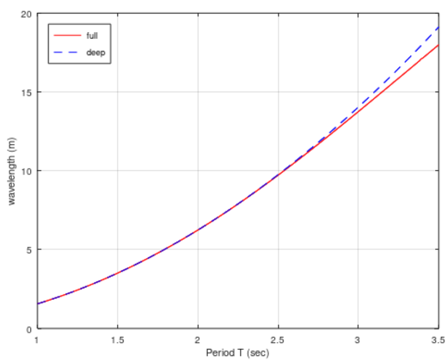

Figure 6.

This is a very useful plot since it gives us information about the wavelength of the waves produced by our bobbing floater. Finally, let's have a look at factor (3) in eq.8, how the added mass depends on the bobbing period. Figure 7 shows both factor (3) and the entire added mass according to eq.8. Plots are shown for water depths 3,4 and 5m.

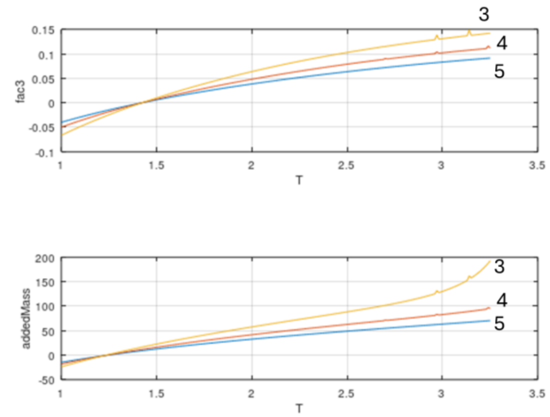

Figure 7.

Factor three increases with period and decreases with depth. But strangely, for small periods it actually goes negative! This shows that added mass cannot be thought of as displacing a mass of water and suggests a resonance phenomenon. The entire mass behaves the same, though the effects of factor (2) on water depth are significant, especially for large periods.

Let's put all of this into context. We wish to design an experiment where the bob will oscillate with a period close to 1.0 sec. Let's start by ignoring the added mass. Using eq.6 we find the required draft is 0.2485m and we shall set this to 0.25 resulting in a period of 1.003 sec. Choosing a radius 0.565m we find the mass of the bob is 258kg and the added mass is calculated at -14.39kg. The period of oscillation is

$$T = 2\pi\sqrt{\left(1+\frac{m_a}{m}\right)\frac{d}{g}} \qquad (11)$$

which is lowered to 0.98 sec. This is observed in simulation.

X.2.3 Radiation Damping

So far, we have been silent about damping. There are two factors which contribute to damping. The first is *radiation damping* which is due to the bob creating waves which radiate out from the bob, transport energy from the bob and therefore contribute to damping. This is clearly visible in the simulation snapshot shown in Fig.8.

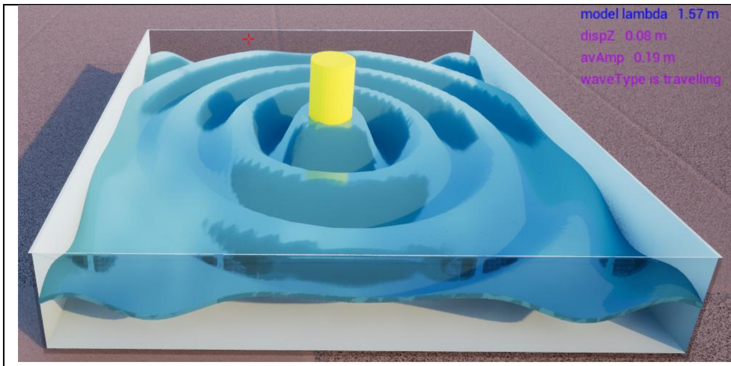

Figure 8. PhysLab wave tank showing radiated waves.

Hydrodynamic theory shows that the radiation damping force is proportional to bob vertical velocity, so we extend the equation of motion,

$$(m + m_a)\ddot{z} = -\rho g A z - b_R \dot{z} \qquad (12)$$

where b_R is the radiation damping coefficient. It will not come as a surprise to learn that the theoretical expression for this coefficient is complicated. Like the expression for added mass, it depends on the water depth and the wavelength, but not on the cylinder draft. The latter is easy to understand, wave generation is essentially a *surface* phenomenon. It is also important to realize that these travelling waves radiate out from the bob and so can be called a *far field* effect. Here's the expression for b_R, again colour coded so we can unpick the contribution of the various terms.

$$b_r = \underbrace{\rho \pi a^3 \omega}_{(1)} \underbrace{\frac{\pi a}{4h}}_{(2)} \underbrace{\frac{1}{\left\{1 + \left(\frac{2\pi a}{\lambda}\right)^2 \left[\frac{\pi}{2} - \gamma - \ln\left(\pi \frac{a}{\lambda}\right)\right]\right\}}}_{(3)} \qquad (13)$$

The first term depends on the oscillation frequency ω and therefore its period. The effect of water depth is confined to term (2) and the third term involves wavelength λ and therefore the period of the bob. Figure 9 shows how terms (2) and (3) vary with period and also the full expression for b_R.

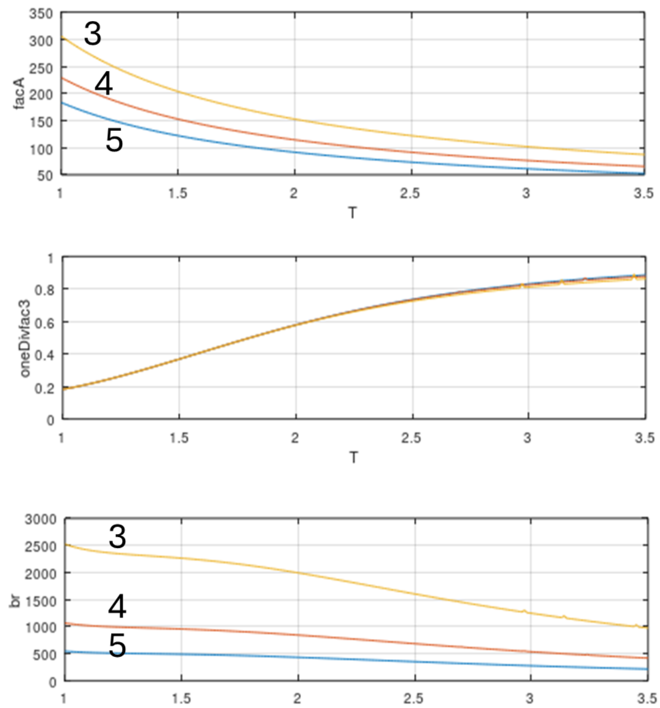

Figure 9

Looking at b_R you can see that this decreases with increasing period and also decreases with increasing water depth. Factor (2) has a large influence as the frequency decreases with period and factor(3) increases with period an shows little variation with depth.

A simulation plot showing the effect of radiation damping is presented in Fig10. Simulation parameters are: draught = 0.25m, mass = 258.25kg, added mass -14.39 kg with radiation damping 61.26 kg/s.

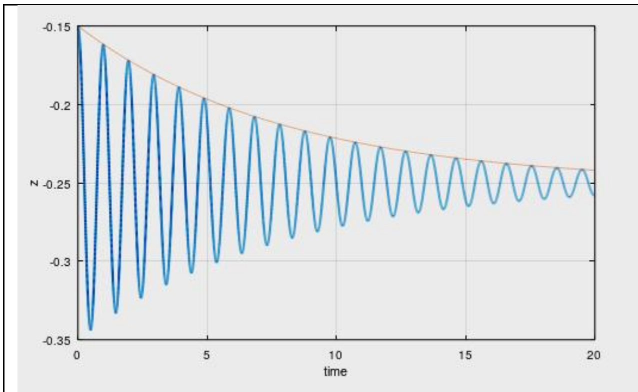

Figure 10.

X.2.4 Viscous Damping

This is due to both the fluid friction along the side of the cylinder where we assume the fluid flow is smooth ('laminar') and perhaps more importantly due to vortex shedding at the flat cylinder bottom. Laminar friction is modelled as a linear term $\sim -\dot{z}$ but vortex shedding is nonlinear and is modelled as $\sim -\dot{z}|\dot{z}|$. This prohibits any sensible analysis, so the effects of viscous damping are studied through experimentation. The expression for the viscous damping coefficient is

$$b_V = \frac{1}{2}\rho C_d \pi r^2 \qquad (14)$$

where C_d is an experimentally determine coefficient, whose value is around 1.0.

It's useful to compare radiation and viscous damping, Fig.11 shows the radiation damping and viscous damping forces as a function of water depth. The horizontal line is the viscous force, and the blue curve is the radiation force calculated using eq.13 Circles show simulation data points. Clearly viscous damping dominates for all except fairly shallow water.

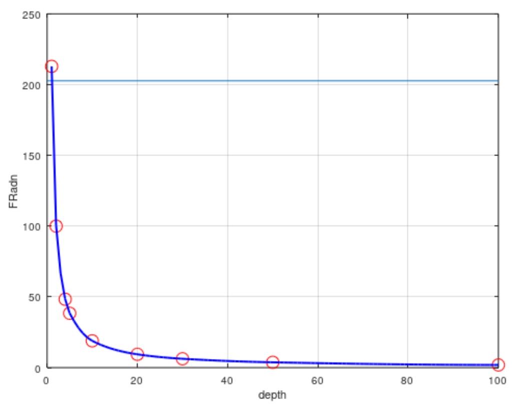

Figure 11.

X.3 Simple Air Spring

Air spring suspension is very common these days, on trucks, trains and cars, so it's useful to understand some physics behind these. Here we take a very simple model, a plunger sitting in a cylinder containing some gas under pressure. Figure 12 shows what we have in mind; in (a) the plunger is suspended in the cylinder of crosssectional area A and traps a volume V_0 of air at pressure p_0 and in (b) we release the plunder which drops a little compressing the air until it comes to equilibrium. This is our 'simple air spring' (SAS).

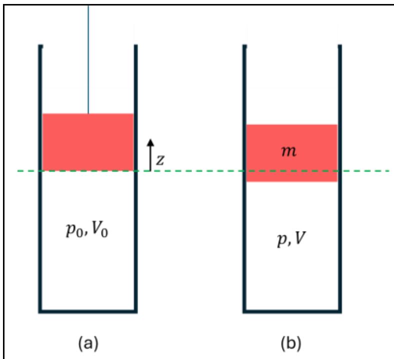

Figure 12.

The equation of motion of the plunger is straightforward,

$$m\ddot{z} = pA - mg - b\dot{z} \qquad (15)$$

where we assume some non-viscous damping. The air will behave almost adiabatically, so we write

$$pV^{\gamma} = p_0 V_0^{\gamma} \qquad (16)$$

and we know that volume and displacement are linked

$$V = V_0 + Az \qquad (17)$$

From eqs16,17 we find an expression for the force on the plunger due to the compressed air, as a function of displacement

$$F(z) = Ap_0 \left(\frac{1}{1 + z/L_0}\right)^{\gamma} \qquad (18)$$

where we have introduced $L_0 = V_0/A$, the initial length of the air column in Fig.12(a). Adopting the parameter set $p_0 = 10, V_0 = 0.5, A = 0.5$ we plot eq.18 in Fig.13 You can easily see how the force is behaving, for upward displacements the volume increases and the pressure drops and so does the force. For downward displacements the volume reduces so both pressure and force increase.

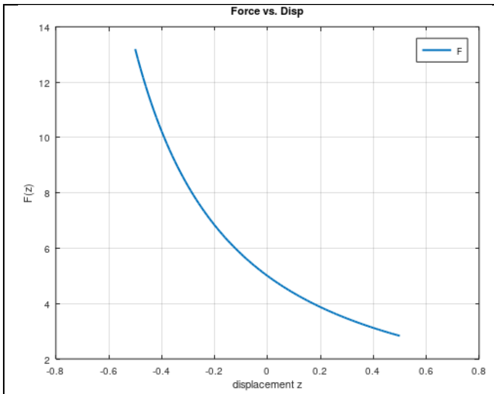

Figure 13.

It seems reasonable to agree that the mass will oscillate about its equilibrium, and to calculate the frequency of low-amplitude oscillations we need to find the stiffness k of the air 'spring'. From eq.16 we find

$$k(p, V) = \frac{dF}{dz} = -\gamma \frac{pA^2}{V} \qquad (19)$$

Fig.14 show how this varies with displacement z. The first thing to note is that the stiffness is always negative which we need for oscillations. As the mass is displaced downwards the stiffness increases in magnitude, and so does its rate of increase. In terms of vehicle suspension this is quite interesting, since the quality of ride will depend on the vehicle load unlike simple coil springs which have a constant stiffness.

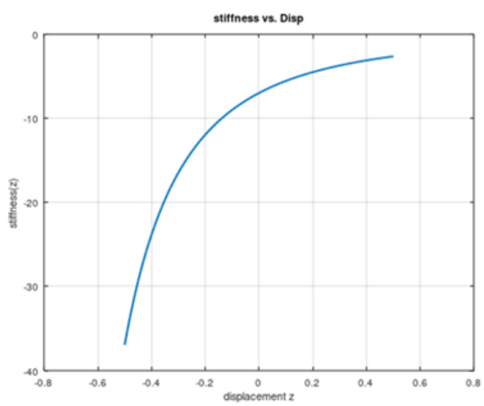

Figure 14.

These oscillations will occur about the equilibrium position of the mass which will vary with m. So we need to understand this position. When the air is supporting the mass, we have $pA - mg = 0$ and using eq.16 after a little algebra, we find the equilibrium displacement of the mass is just

$$z_{Equ}(m) = L_0\left[\left(\frac{p_0 A}{mg}\right)^{1/\gamma} - 1\right] \qquad (20)$$

This makes sense; as *m* is made large then $z_{Equ} \to -L_0$ so the plug is near the bottom of the tube. Also if we wish $z_{Equ} = 0$ then the mass must obey the relation $m = p_0 A/g$. At equilibrium we have the following pressure

$$p_{Equ}(m) = \frac{mg}{A} \qquad (21)$$

and using eq.16 the volume at equilibrium is just

$$V_{Equ} = \left(\frac{p_0 A}{mg}\right)^{1/\gamma} V_0 \qquad (22)$$

Inserting the latter into our expression for stiffness, eq.19 we find an expression for stiffness as a function of mass *m* at equilibrium.

$$k_{Equ}(m) = -\gamma\frac{mg}{L_0}\left(\frac{mg}{p_0 A}\right)^{\frac{1}{\gamma}} \qquad (23)$$

and therefore for frequency

$$\omega_{Equ}^2(m) = -\gamma\frac{g}{L_0}\left(\frac{mg}{p_0 A}\right)^{\frac{1}{\gamma}} \qquad (24)$$

Now this tells us something interesting, that the frequency *increases* with mass. We are not used to this, e.g. for a mass on a spring we have $\omega^2 = k/m$. You can see why it's so, when we increase the mass the gas volume decreases and its pressure increases, so from eq.19 we see that the 'air spring' stiffness increases.

Bringing all of this together, we shall explore how to plan an investigation into the effects of mass *m* on the position and frequency of low-amplitude oscillations. We start with eq.20 and plot the equilibrium position and from eq.24 we plot the period of oscillation, both as a function of *m*. Results are shown in Fig.15 together with some data points from a subsequent simulation.

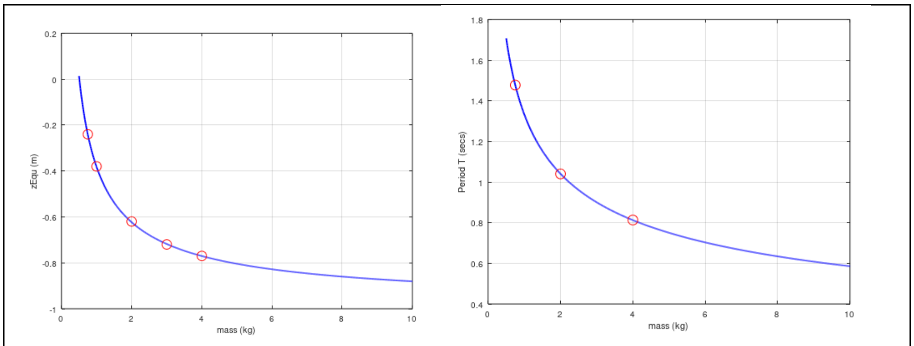

Figure 15.

X.4 Rotating Air Spring

One can imagine an engineering situation where our SAS could be located in a rotating situation, perhaps some sort of engine. A cylinder, open at one end, with a plunger is fixed to the centre of a rotating platform as shown in Fig.16 with air on both sides.

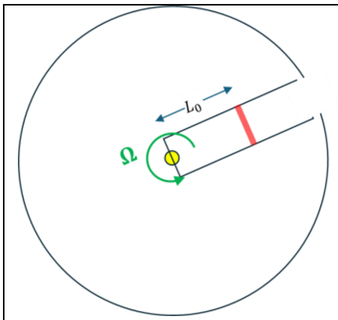

Figure 16.

When the platform is not rotating, the pressures on either side of the plunger are the same, p_0, the length of the confined air is L_0 and the volume if this air is $V_0 = AL_0$ where A is the area of the cylinder. This is detailed in Fig.17(a).

When the platform is rotating with angular speed Ω then the centrifugal force $m\Omega^2 a$ pushes the plunger outwards, so the pressure in the confined part is reduced to p. This is detailed in Fig.17(b).

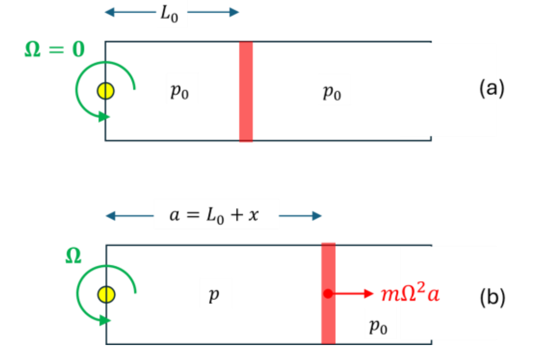

Figure 17.

Clearly the confined air is acting as a nonlinear spring resisting the centrifugal force on the plunger. The question is, how well can the force of the confined air perform against the centrifugal force? We can imagine that for a small Ω the air spring would hold its own, and there could be a stable location a. But could there be a critical Ω where the air spring cannot constrain the plunger which would then fly out of the tube into inexistence? Also we need to think about oscillations, can this system oscillate, and how could Ω influence its frequency if there is one?

So many questions, and we need to go on a short maths journey towards the answers. Let's start with the equation of motion of the plunger (which we have assumed is not thick). Using the notation from Fig.17 we have

$$m\ddot{x} = (p - p_0)A + m\Omega^2 a - b\dot{x} \qquad (25)$$

where we have included notional damping. We also need to remember how gas pressures (and therefore forces) behave. From the adiabatic law $pV^\gamma = p_0 V_0^\gamma$ we have

$$p = p_0\left(\frac{L_0}{a}\right)^\gamma \qquad (26)$$

So we have a starting place. Perhaps the next thing to do is to look an expression for the force on the plunger as a function of its position a. From the above this is

$$F(\Omega, a) = p_0\left(\frac{L_0}{a}\right)^\gamma A - p_0 A + m\Omega^2 a \qquad (27)$$

This is a somewhat annoying expression; the force is a function of both Ω and a and a occurs in two places. It must be solved numerically but a good understanding can be obtained by plotting the force as a function of a for various values of Ω, see Fig.18.

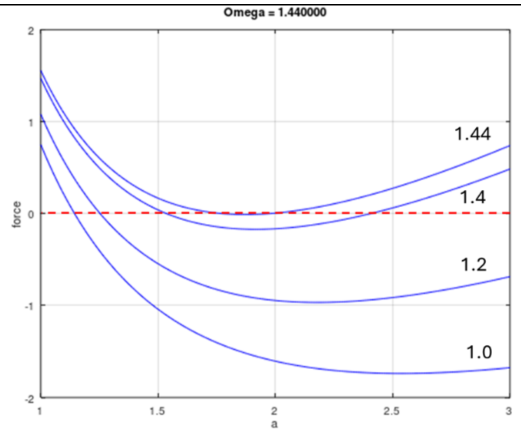

Figure 18.

You can see that as Ω is increased from zero, there are two values of a where the force is zero. One of these equilibrium points is stable (negative derivative) and the other is unstable. Eventually there is one value of a where there is just one equilibrium point, here the force and its derivative are both zero.

The derivative is just

$$\frac{dF}{da} = (1+\gamma)m\Omega^2 - \frac{p_0 A\gamma}{a} \qquad (28)$$

so setting this and the expression for force both to zero, we obtain an expression for the critical Ω where the equilibrium point just disappears,

$$\Omega_{crit}^2 = \frac{p_0 A\gamma}{mL_0}(1+\gamma)^{-(1+1/\gamma)} \qquad (29)$$

Finally, from eq.28 we find the frequency of oscillation of the plunger

$$\omega^2 = \frac{p_0 A\gamma}{ma} - (1+\gamma)\Omega^2 \qquad (30)$$

which tells us that the rotation reduces the frequency from its natural frequency of $p_0 A\gamma/ma$ to zero which occurs at a second critical rotation speed

$$\Omega^2 = \frac{p_0 A\gamma}{ma(1+\gamma)} \qquad (31)$$

This is of limited use since it is a function of a, nevertheless when equilibrium values are available from simulations, it allows the checking of observed plunger frequencies, see Fig.19.

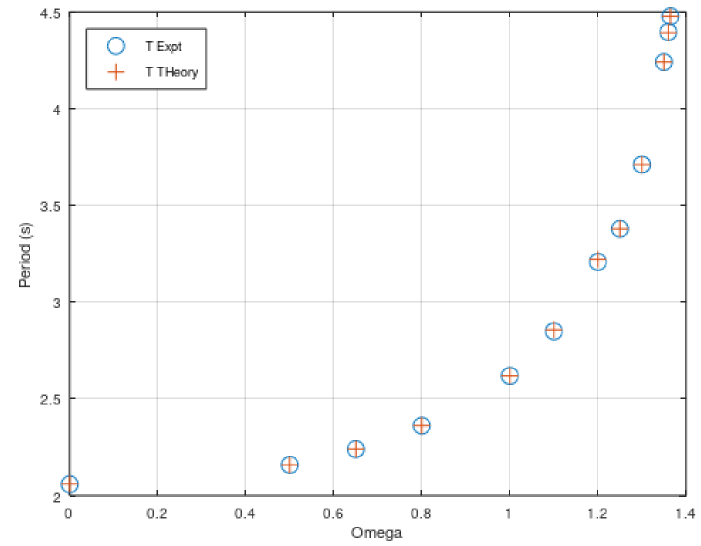

Figure 19.

Simulation parameters were, $L_0 = 1, p_0 = 10, m = 0.75, A = 0.5$. Agreement is good right up to Ω_{Crit} which from eq.29 is evaluated as 1.443.