Wind Turbine Control UPDATED 25-04-21

This document is very much a work in progress. While it was originally intended to support Comp3352, it is growing beyond that (since 3352 ceases to exist after this run). I hope that a Computing Project student will take this on as a project next year. Stuff directly relevant to Comp3352 is indicated Comp3352

1. Introduction Comp3352

The most common form of turbines are horizontal axis such as the CART3[1] experimental turbine shown in Figure 1.

Figure 1.

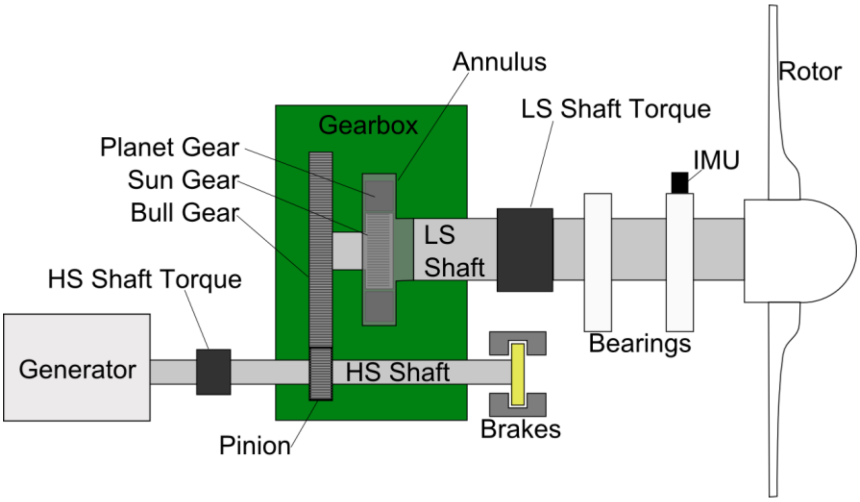


Figure 2.

This shows the main components: the tower, the rotor and the nacelle. The latter houses the low-speed shaft connected to the rotor, an up-speeding gearbox connected via a high-speed shaft to the electricity generator. This is illustrated in Figure2. The available wind energy is partially captured by the rotor where it appears as kinetic energy of rotation which then is converted to electrical energy. The power generated depends on the wind speed. Figure 3 (left) shows a graph of the power generated by the CART3 turbine as a function of wind speed experienced. There are three clear "regions" of operation. In Region 1, the wind speed is below 2.5 m/s and at these speeds the power generated is less than the loss of power in the machinery, so the turbine is not allowed to rotate. In Region 2 where the wind speed is between 2.5 and 11.7 m/s the turbine adapts its speed to extract the largest amount of power. Region 3 is associated with wind speeds from 11.7 to 20 m/s. Here the power absorbed and generated is limited to the operating point to ensure safe mechanical and electrical load limits. This can be done by changing the pitch of the rotor blades so they extract less wind energy, or by applying mechanical brakes. Finally, if the wind speed should exceed 20 m/s then the control systems may not cope, so the turbine is shut down.

[1] Details obtained from the National Renewable Energy Laboratory (Colorado, US) and from personal conversations with Ervin Bossanyi, Principal Engineer, Turbine Engineering, GL Garrad Hassan, Bristol UK.

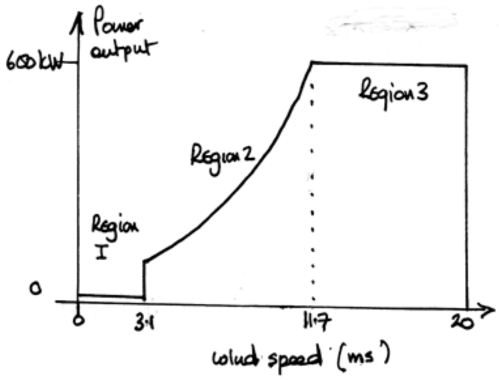
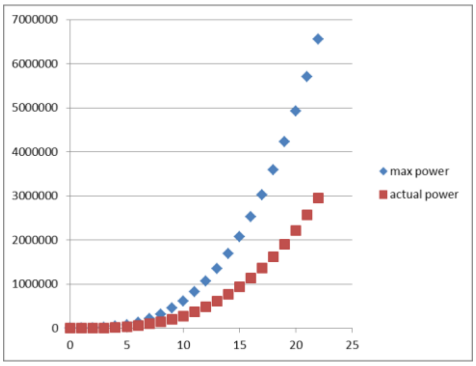


Figure 3.

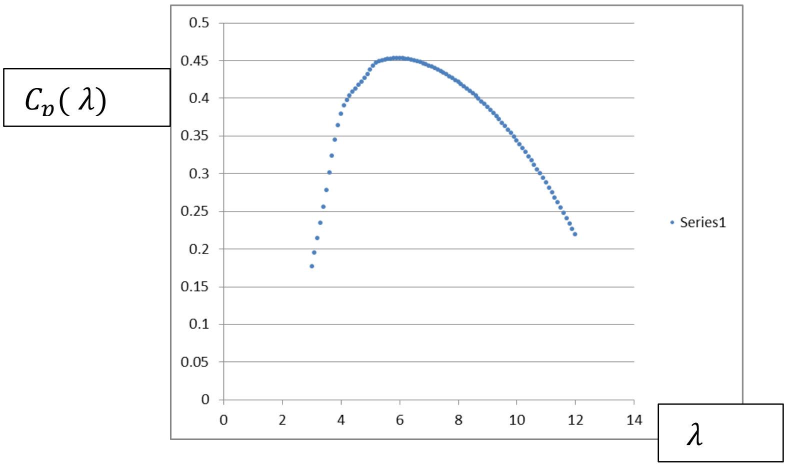


The maximum amount of power which can be absorbed from the wind is not 100% of the available wind power, rather it is only 59% as shown by Betz. Imagine if the turbine absorbed 100% of the wind energy, that would mean that the wind leaving the rotor would have no kinetic energy, is it would stop! This would prevent any more wind from passing through the rotor, so this can't happen. The theory behind this will be explained later.

The amount of power that can be absorbed from the wind is

$$P = \frac{1}{2}\rho\pi R^2 C_p v^3 \qquad (1a)$$

where R is the radius of the rotor. The power depends on the velocity cubed (last term in the expression). But it also depends on the efficiency of power capture. This is known as the turbine's power coefficient C_p which is established by measurement; it varies as shown in the diagram below. You can see that this has a maximum of around 0.45, so the efficiency is 45%. The turbine controller must ensure that the turbine operates at this point. The curve $C_p(\lambda)$ is expressed as a function of λ which is the ratio of the speed of the tip of the turbine blade (in m/s) to the speed of the wind, also in m/s.



The variable λ is known as the "tip-speed ratio" and is defined as

$$\lambda = \frac{\omega R}{v_{wind}} \qquad (1b)$$

where ω is the angular speed of the rotating turbine.

When wind passes through the blades causing them to rotate, there is a torque applied to the turbine shaft. This is given by

$$\tau_{wind} = \frac{1}{2}\rho\pi R^5 \frac{C(\lambda)}{\lambda^3}\omega^2$$

So you can see that the maximum power and torque are obtained for an optimal tip-speed ratio which places $C(\lambda)$ at the maximum of its curve as discussed above.

2. Region 2 Controller. The Basic Controller Comp3352

So as the wind speed changes, the turbine controller must adjust the turbine rotational speed ω so it rotates with the optimal tip-speed ratio.

The controller does this by using the generator which applies an opposite torque to the turbine shaft. The generator torque is arranged to be the following

$$\tau_{gen} = \frac{1}{2}\rho\pi R^5 \frac{C_{pMax}}{\lambda_{opt}^3}\omega^2$$

where you can see it will provide optimal power extraction. Of course the torque provided by the wind will not equal that, we have an error $\tau_{wind} - \tau_{gen}$ which needs to go to zero. The difference in torques will provide an angular acceleration to the shaft connecting turbine and generator, so its angular velocity will change according to the following expression (where *j* is the "moment of inertia" of the rotating components, playing the role of mass in Newton's second law *a = F/m*)

$$\Delta\omega = \frac{(\tau_{wind} - \tau_{gen})}{J}\Delta t$$

This changes the rotational speeds until the torques are equal. Putting this all together we find

$$\Delta\omega = \frac{1}{2J}\rho\pi R^3 \left(\frac{C(\lambda)}{\lambda^3} - \frac{C_{pMax}}{\lambda_{opt}^3}\right)\omega^2$$

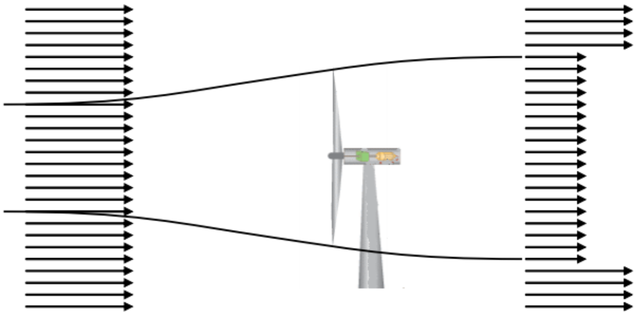
So when $\lambda > \lambda_{opt}$ (the turbine is turning too fast), then $C(\lambda)$ is too small (see the above graph) so the first term in the brackets is smaller than the second, so $\Delta\omega < 0$ and the turbine slows down. A similar argument follows when the turbine is rotating too slowly, then $\Delta\omega > 0$ and the turbine will speed up.

This form the basis of the controller you will code

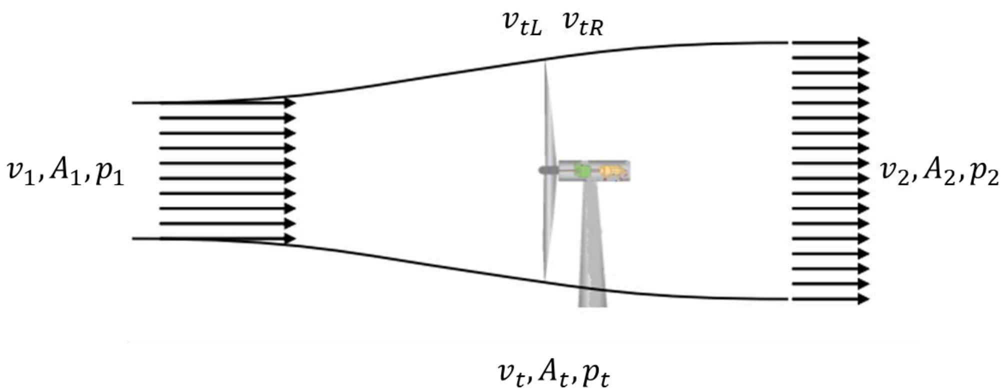
3. Derivation of the Wind Power and Torque Equations and the Betz Limit

NEW 21-04-21

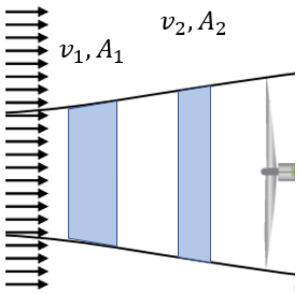
Let's have a look at wind passing a single turbine. In the diagram below arrows show the wind velocity. Clearly the turbine reduces wind velocity as it extracts power. That should be obvious.



Let’s consider the wind passing through a streamtube, the upstream wind velocity is v_1, the downstream velocity is v_2 and the velocity at the turbine is v_t.



The basic idea is all air entering the left of the streamtube stays in the streamtube until it exits the streamtube on the right. The entry parameters, (velocity, tube area and pressure), are v_1, A_1, p_1, similar tuples are shown near the turbine and on the entry of the tube. Clearly the area of the tube increases, buw what about the wind velocity. The following diagram will help



The blue areas show a given element of volume as it moves through the tube. Assuming air is incompressible then the volume stays the same. Therefore the length of the element must get smaller since the area gets larger. So we have for the volumes

$$\Delta l_1 A_1 = \Delta l_2 A_2$$

But the above snapshot shows movement over the same time interval so dividing the above by this interval we find how the volume changes with time

$$\frac{\Delta l_1}{\Delta t} A_1 = \frac{\Delta l_2}{\Delta t} A_2 = v_1 A_1 = v_2 A_2$$

So we see that the air slows down. Multiplying by the air density ρ we get an expression for the rate of change of mass (the same at all places)

$$\frac{\Delta m}{\Delta t} = \rho A_1 v_1 = \rho A_t v_t = \rho A_2 v_2$$

Hence the rate of change in momentum of the air stream between when it enters and when it leaves the streamtube is

$$\frac{\Delta m}{\Delta t} v_1 - \frac{\Delta m}{\Delta t} v_2 = \rho A_t v_t (v_1 - v_2)$$

and this is caused by the thrust on the turbine disc.

We can use Bernoulli on in the streamtube at the left and right sides of the turbine disc

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_{tL} + \tfrac{1}{2}\rho v_{tL}^2$$

$$p_{tR} + \tfrac{1}{2}\rho v_{tR}^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

Assuming that $p_1 = p_2$ and $V_{tL} = V_{tR}$ then the pressure difference across the disc is

$$p_{tL} - p_{tR} = \tfrac{1}{2}\rho (v_1^2 - v_2^2)$$

Since force is pressure times area, we can use this to get another expression for force on the disc

$$F = \tfrac{1}{2}\rho (v_1^2 - v_2^2) A_t$$

Equating the two expressions for force we find

$$\rho A_t v_t (v_1 - v_2) = \frac{1}{2}\rho (v_1^2 - v_2^2) A_t$$

$$v_t = \frac{1}{2}(v_1 + v_2)$$

which tells us the windspeed at the turbine is the average of the upwind and downwind wind speeds.

Finally, we have an expression for the power delivered to the turbine. Power is force x velocity, so at the turbine we find

$$P = F_t v_t = 2\rho A_t v_t^2 (v_1 - v_t)$$

This is an interesting expression which shows how the power depends on the wind speed near the turbine. There are two factors, v_t^3which increases with v_t and $(v_1 - v_t)$ which decreases with v_t, so we expect the power - $v_t$curve to have a peak which it does. To find the peak we proceed as usual,

$$\frac{\partial P}{\partial v_t} = 0,$$

which gives the result $v_t = \frac{2}{3} v_1$ and also $v_2 = \frac{1}{3} v_1$.

Finally we turn to the maximum power expression, plugging this value of v_t into the expression for power *P* we find

$$P_{max} = \frac{8}{27}\rho A_t v_1^3 \qquad (2)$$

and if we define the power coefficient as

$$C_p = \frac{P}{\frac{1}{2}\rho A_t v_1^3}$$

then we find

$$C_{p,max} = \frac{16}{27} = 0.593 \qquad (3)$$

This is the maximum achievable efficiency of a wind turbine (59.3%) and is known as the Betz limit. Note how the maximum of C_p for the CART-3 turbine (shown in the initial section above) is less than this.

4. Improvements to Region 2 Control Comp3352

See papers by Johnson and Fingersh. One idea is "Optimally Tracking Rotor Control" which resulted from the observation that the turbine did not respond rapidly to changes in wind speed because of its inertia. This means that it was unable to accelerate fast enough. To fix this, the torque provided by the generator is adjusted with a different control law.

Writing with $K = \frac{1}{2}\rho\pi R^5 \frac{C_{pMax}}{\lambda_{opt}^3}$ we can re-write the original equation for turbine torque as

$$\tau_{gen} = K\omega^2$$

so the original control law becomes

$$\Delta\omega = \frac{1}{J}(\tau_{wind} - K\omega^2)\Delta t$$

Then if the turbine angular speed is too small, then the bracket is positive and the turbine will speed up and vice-versa. In the revised system, the generator torque is calculated according to the following expression, where an extra term is added

$$\tau_{gen} = K\omega^2 - G(\tau_{wind} - K\omega^2)$$

The "gain" G lies between 0 and 1. To see the effect of this extra term, consider the new control law

$$\Delta\omega = \frac{1}{J}[\tau_{wind}(G+1) - K\omega^2(1-G)]\Delta t$$

The first term is now larger, so the controller responds more rapidly to changes in the wind. The second term is smaller (and approaches 0 as G approaches 1)

5. Inclusion of Gearbox and damping.

The expression for the rotation of the turbine shaft now becomes

$$\Delta\omega = \frac{\left(\tau_{wind} - n\tau_{gen} - \left(b_r + n^2 b_g\right)\omega\right)}{\left(J_r + n^2 J_g\right)}\Delta t$$

where subscripts refer to rotor and generator respectively, *n* is the gearbox ratio and *b* refers to friction coefficients. The control law then uses

$$\tau_{gen} = \frac{K\omega^2}{n}$$

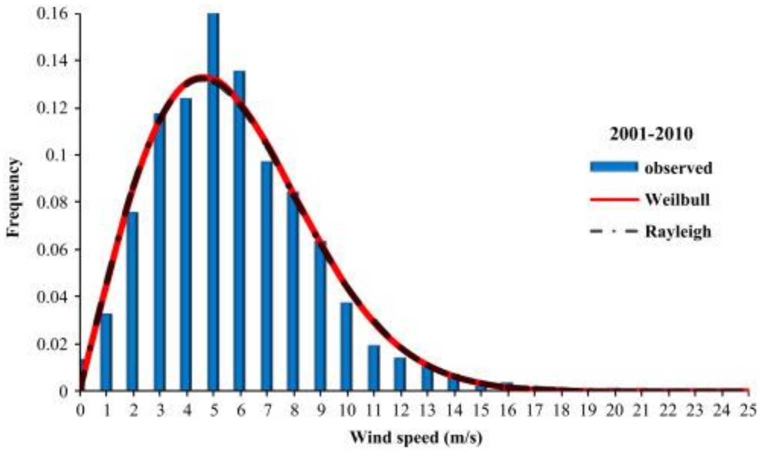
Interestingly, of we neglect the friction, the only effect of the gearbox on the controlled system is to magnify the contribution of the generator inertia; in essence we return to our starting controller equation.

For CART 3 we have

Rotor Radius	R	20m
air density	ρ	1.20
rotor inertia	J_r	3.25e5
generator inertia	J_g	34.4
rotor friction	b_r	27.36
generator friction	b_g	0.2
gear ratio	n	43.165
cut in wind speed		5.0
rated wind speed		11.7
cut our speed		22

6. Simulating the Wind Comp3352

Wind speed fluctuates and these fluctuations are best described statistically through the distribution function of speeds. A typical distribution looks like this



Such distributions are successfully modelled using the Weibull or Rayleigh distribution functions. The Rayleigh is a little easier (it is a special case of the more general Weibull) and both have been fitted to the data above. The Rayleigh distribution looks like this

$$f(v) = \frac{2v}{c^2} e^{-(v/c)^2}$$

where the constant c is related to the average wind speed and its variance

$$\mu = \frac{c\sqrt{\pi}}{2} \qquad \sigma^2 = c^2\left(2 - \frac{\pi}{4}\right)$$

We wish to construct a ***time series*** of wind speeds which conforms to this distribution function. It turns out by a remarkable theorem, that we can use the inverse of the associated cumulative distribution function. In this case we have

$$v = c\sqrt{-ln(n)}$$

where *n* is a uniformly distributed random variable between 0 and 1. Simply coding this equation and feeding it with random numbers will produce the required time series of wind speeds.

7. Automating Data Collection Comp3352 NEW 20-04-21

Often we want to vary wind speed and measure something like power output. Sure we could use **setParams** but there's an easier way.

i) First create a new timer in **Actor_Initialize(…)**

```
 if(bApplySteppedWindSpeeds)
   setTimer(stepWindSpeedInterval,true,'changeWindspeedTimer');
```

ii) Now create the call-back function

```
function changeWindspeedTimer() {
 if(windV < vCutout)
   windV += stepWindSpeed;
}
```

iii) Declare the variables up top

```
var(MAS14) float stepWindSpeed;
var(MAS14) float stepWindSpeedInterval;
var(MAS14) bool bApplySteppedWindSpeeds;
```

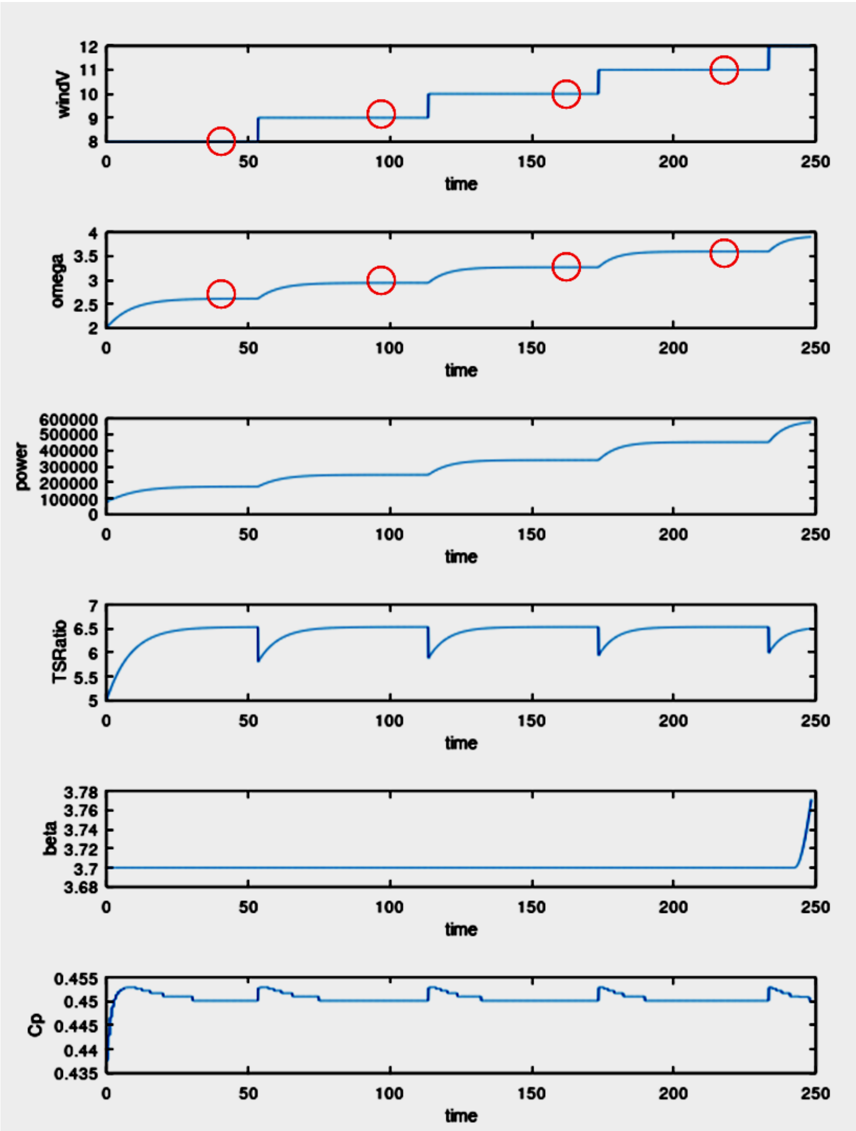
iv) Now set some default values at the bottom in **defaultproperties**

```
 bApplySteppedWindSpeeds=false
 stepWindSpeedInterval=50.0
 stepWindSpeed=2.0
```

This will allow you to create some nice plots like the one shown below

So you have windV at the top, then omega and power, and some other stuff. Don't worry about *beta;* that is for region-III where the blade pitch changes.

You could grab the data shown by the red circles and use this to plot graphs of how various quantities vary with windspeed. Or you could look at the data matrix in the Octave file directly.



If you want to plot several graphs like this using Octave (automatically) then add a function like this into your code.

Note the correspondence between the array indices in the arrays columnLabels and dataArray correspond to the logfile numbering, so you could make the first code chunk cleaner by using a loop.

```
function writeMatlabFileFooterNew(array<String> ccolumnLabels) {
 local int i;
 local String str;

 mlog.logF("];");
 i=0;
 while(i < ccolumnLabels.length) {
  str = ccolumnLabels[i]@"=A(:,"@i+1@");";
        mlog.LogF(str);
  i++;
 }

 // windV
 mlog.logF("subplot(5,1,1);");
 mlog.logF("plot("@ccolumnLabels[0]@","@ccolumnLabels[1]@");");
 str = "xlabel('"@ccolumnLabels[0]@"')";
 mlog.Logf(str);
 str = "ylabel('"@ccolumnLabels[1]@"')";
 mlog.Logf(str);

 // omega
 mlog.logF("subplot(5,1,2);");
 mlog.logF("plot("@ccolumnLabels[0]@","@ccolumnLabels[2]@");");
 str = "xlabel('"@ccolumnLabels[0]@"')";
 mlog.Logf(str);
 str = "ylabel('"@ccolumnLabels[2]@"')";
 mlog.Logf(str);

 // power
 mlog.logF("subplot(5,1,3);");
 mlog.logF("plot("@ccolumnLabels[0]@","@ccolumnLabels[3]@");");
 str = "xlabel('"@ccolumnLabels[0]@"')";
 mlog.Logf(str);
 str = "ylabel('"@ccolumnLabels[3]@"')";
 mlog.Logf(str);

 // tsr
 mlog.logF("subplot(5,1,4);");
 mlog.logF("plot("@ccolumnLabels[0]@","@ccolumnLabels[4]@");");
 str = "xlabel('"@ccolumnLabels[0]@"')";
 mlog.Logf(str);
 str = "ylabel('"@ccolumnLabels[4]@"')";
 mlog.Logf(str);

 // Cp
 mlog.logF("subplot(5,1,5);");
 mlog.logF("plot("@ccolumnLabels[0]@","@ccolumnLabels[5]@");");
 str = "xlabel('"@ccolumnLabels[0]@"')";
 mlog.Logf(str);
 str = "ylabel('"@ccolumnLabels[5]@"')";
 mlog.Logf(str);
}
```

Of course, you will need to make sure you log the data in these functions

```
function setLogFileColumnLabels() {
  columnLabels.length = 0; //empty array.

  columnLabels[0] = "time";
  columnLabels[1] = "windV";
  columnLabels[2] = "omega";
  columnLabels[3] = "power";
  columnLabels[4] = "TSRatio";
  columnLabels[5] = "Cp";
  writeMatlabFileHeaderNew(columnLabels);
}
```

and

```
function logDataRecord() {
  local array<float> dataArray;

  dataArray.length = 0; //empty array.

  dataArray[0] = time;
  dataArray[1] = windV;
  dataArray[2] = omega;
  dataArray[3] = power;
  dataArray[4] = tsr;
  dataArray[5] = Cp;

  writeMatlabFileRecordNew(dataArray);

}
```

8. Region-III controller Coming Soon

9. Some Investigations for a Single Turbine Comp3352 NEW 22-04-21

Most students in the past have done around 3 of the suggestions below.

1) Perhaps the first thing to do is to establish the power versus wind speed curve, at least for region-II and compare this with the Betz limit.

2) Look at the power coefficient at various wind speeds and compare with the theoretical maximum (Betz_

3) Look at the response to step increases in wind speed, especially how the tip-speed ratio and power coefficient change.

4) Investigate changing the radius of the blade. This is a useful design procedure in the industry where the characteristics of an operating turbine can be used to estimate the characteristics of a larger turbine. Here is how various quantities 'scale'. You could test these empirically.

- Power is proportional to the area swept out by the blades, so proportional to the radius squared.
- At a constant tip speed ratio, doubling the radius will halve the rotor speed
- When the radius is doubled the torque will be proportional to the cube of the radius, since power is quadrupled and rotor speed will be halved.

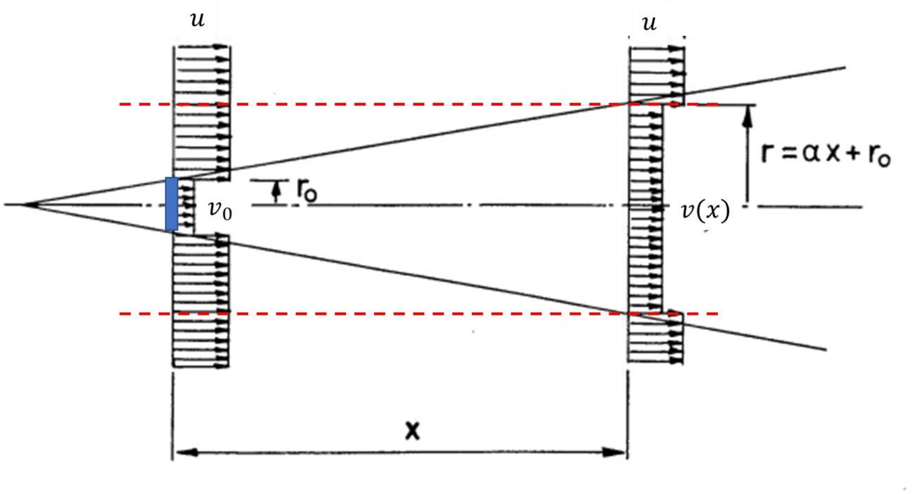
5) Investigate response to varying wind speeds. Use the average wind speed as the independent variable.

10. Simulating Wind Farms NEW 22-04-21

This is perhaps more suitable for a Computing Project rather than this module assignment.

10.1 Wake behind a single turbine

The sketch below is adapted from Jensen's paper and shows the wake developing behind a single turbine, shown as the blue rectangle. The wind speed *u* is the ambient wind speed at this location. The turbine extracts energy from the wind and thereby reduces its speed, as shown, to v_0. As this wind spreads out in the wake, its speed will progressively get smaller, at a distance *x* the speed will be $v(x)$. It's this function we need to calculate.



If v_0 is the wind speed in front of the farm, then we know that $v_0 = \frac{1}{3}u$. Consider the air arriving at the section on the right, a distance *x* from the left turbine. This air comes from a column delineated by the dotted red lines. There are two contributions, first from the cone emerging from the left turbine

$$\pi r_0^2 v_0$$

and second from the column minus the cone

$$\pi(r^2 - r_0^2)u$$

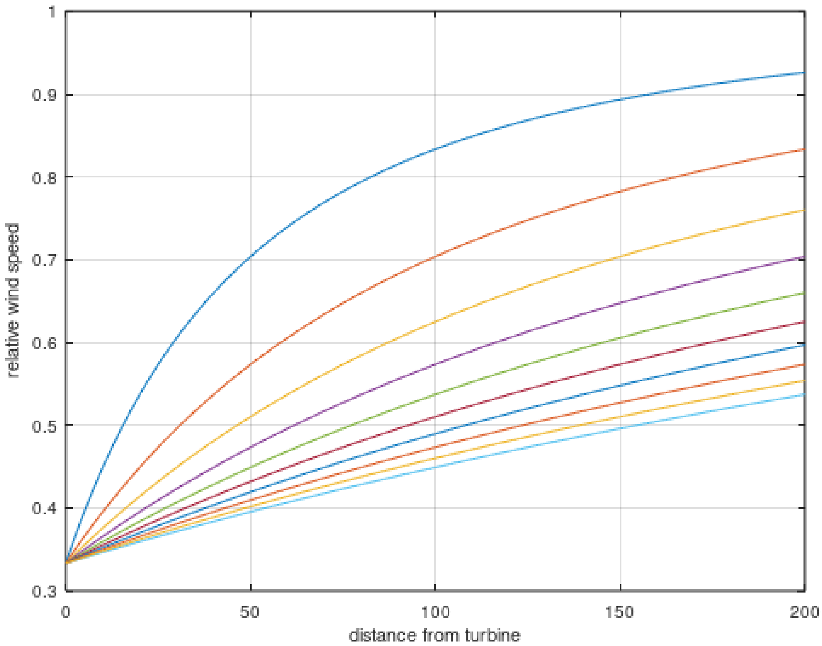
We therefore have

$$\pi r_0^2 v_0 + \pi(r^2 - r_0^2)u = \pi r(x)^2 v(x) \qquad (10.1)$$

Using the fact that $v_0 = \frac{1}{3}u$ we can solve for *v(x)* as a function of *x*.

$$v = u\left[1 - \frac{2}{3}\left(\frac{r_0}{r_0 + \alpha x}\right)^2\right], \qquad (10.2)$$

Solutions to this equation for a number of rotor diameters are shown below.



Curves are shown for r_0= 10, 20, 30, … 100, in this order starting from the dark blue top line. These curves show the relative wind speed *v/u*. Note that close to the turbine disc (x=0) we have $v = \frac{1}{3}u$ as indicated by theory. As you move downwind, the windspeed starts to recover, the relative speed marches slowly towards 1. The larger the turbine blade, the slower is this march; larger turbines provide a greater reduction in the windspeed at any downwind point. The coefficient α depends on the surface roughness, for land based farms Jensen suggests this is about 0.1 which corresponds to a wake angle of 5.7 degrees.

We have followed the thinking in Jensen's original paper to derive this result. The later Katic paper extended this result, realizing that the factor 2/3 was too limiting, and the 'velocity' deficit' for a turbine depends on the operating point of the turbine. The idea is that the deficit depends on the thrust applied to the turbine blades. Rather like the power coefficient C_p, the *thrust* coefficient $C_T$is defined as

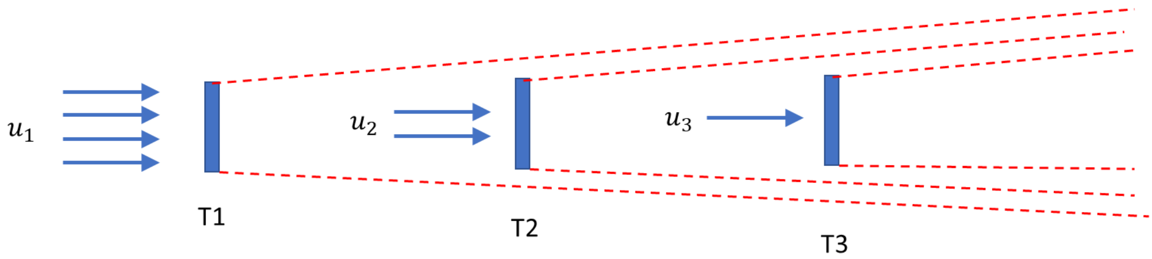
$$C_T = \frac{T}{\frac{1}{2}\rho A_t v_1^2}$$

where *T* is the thrust. Note the velocity squared in the denominator rather than velocity cubed for the power coefficient. The above expression for *v* in terms of *u* then becomes

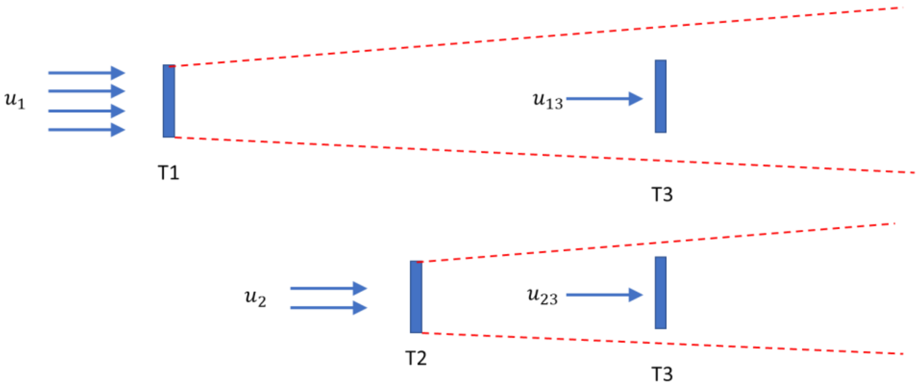
$$v = u\left[1 - \frac{\sqrt{1 - C_T}}{(1 + kx/r_0)^2}\right] \qquad 10.3)$$

10.2 Overlapping Wakes

Here things start to get really interesting. You have seen that the windspeed is reduced in the wake of a turbine, so if a second turbine is located in the wake of a first, it receives a lower windspeed. In turn it also reduces the windspeed in its wake. So how do we do the calculation? Here we follow the Shao et al., paper. Consider the configuration of turbines below where T2 is in the wake of T1 and T3 is in the wake of T1 and T2.



Well we split this up, looking at T3 experiencing the wake from T1, then T3 experiencing the wake from T2 just like this



The actual calculation proceeds by calculating the 'kinetic energy deficit' which is the ratio of the wind kinetic energy lost by the turbine to the kinetic energy put in. Since kinetic energy is proportional to speed squared, such ratios would look like this,

$$\left(\frac{u_{in} - u_{out}}{u_{in}}\right)^2 \equiv \left(1 - \frac{u_{out}}{u_{in}}\right)^2$$

So for this example to calculate the deficit of T1 and T2 into T3 we first calculate the deficit due to T1

$$\left(1 - \frac{u_{13}}{u_1}\right)^2$$

where we calculate u_{13} using equation 10.3. Then we calculate the deficit due to T2

$$\left(1 - \frac{u_{23}}{u_2}\right)^2$$

where u_2 and u_{23} are calculated using equation 10.3. Then we sum the deficits to obtain

$$\left(1 - \frac{u_3}{u_1}\right)^2 = \left(1 - \frac{u_{13}}{u_1}\right)^2 + \left(1 - \frac{u_{23}}{u_2}\right)^2$$

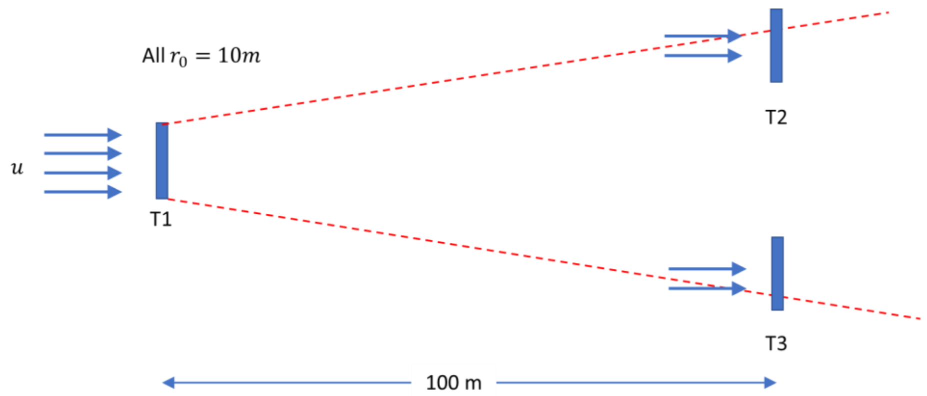
The speed u_1 is the 'input' speed to T1 and the speed u_3 (which we are calculating) is the 'input' speed to T3. The speed u_2 depends on the location *x* of T2, the speeds u_{13} and u_{23} depend on the location of T3. All these can be calculated using equation 10.2 with the correct value of *x* used for each turbine.

We may generalize this calculation to a situation where we have *N* upwind turbines to our 'target' turbine with index *i*:

$$\left(1-\frac{u_i}{u_1}\right)^2 = \sum_{j=1}^{N}\left(1-\frac{u_{ji}}{u_j}\right)^2$$

Let's have a look at a worked example or two.

Example 1.

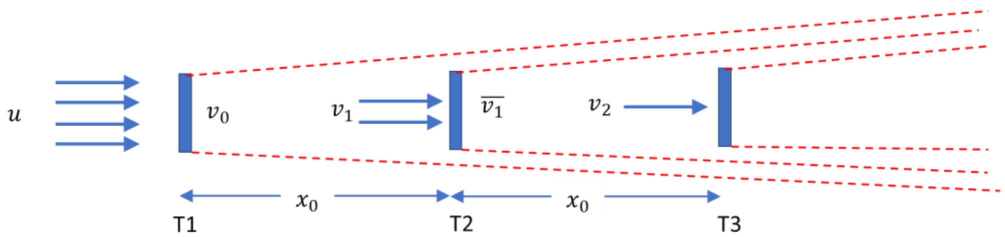


Here we have T2 and T3 downwind of T1 and in the wake of T1. Equation 10.2 tells us that the velocity near T2 and T3 = 0.833u. So the ratio of the power output of this configuration to 3 non-interacting generators is

$$\frac{P}{P_0} = \frac{u^3 + 2\,(0.833u)^3}{3u^3} = 0.796$$

So this configuration is around 80% efficient as non-interacting turbines.

There is another approach which is less complex, and we return to Jensen. Let us consider a column of N turbines sitting in the wakes of upwind turbines. Jensen, as we shall see, simplifies the 'mixing' approach presented above.



Working on the same principle used in deriving 10.1 we calculate the wind speed approaching turbine T2,

$$\pi r_0^2 v_0 + \pi(r^2 - r_0^2)u = \pi r^2 v_1$$

and, solving for $v_1$we have, using $r = r_0 + \alpha x_0$,

$$v_1 = u\left[1 - \frac{2}{3}\left(\frac{r_0}{r_0 + \alpha x_0}\right)^2\right]$$

To find the velocity in front of T3 we apply the same approach giving,

$$\pi r_0^2\left(\frac{1}{3}v_1\right) + \pi(r^2 - r_0^2)\bar{v}_1 = \pi r^2 v_2$$

Here $\bar{v}_1$is the 'mixed' velocity coming in from the wake of T1 and T2 (as discussed above) and Jensen suggests this can be approximated by *u* the incident velocity. So, the above expression becomes,

$$\pi r_0^2\left(\frac{1}{3}v_1\right) + \pi(r^2 - r_0^2)u = \pi r^2 v_2$$

and, solving for v_2

$$v_2 = u\left[1 - \left(1 - \frac{1}{3}\frac{v_1}{u}\right)\left(\frac{r_0}{r_0 + \alpha x_0}\right)^2\right]$$

This relationship holds when we walk down the column of *N* turbines. So, moving from the *N-1*'th to the *N*'th turbine we have

$$v_N = u\left[1 - \left(1 - \frac{1}{3}\frac{v_{N-1}}{u}\right)\left(\frac{r_0}{r_0 + \alpha x_0}\right)^2\right]$$

This recurrence relationship is easy to program (Octave, hehe). For a rotor radius r_0= 10m and a turbine spacing x_0= 50m we have the following wind speed ratios (at the *exit* of the numbered turbine)

i	1	2	3	4	5	6	7	8	9
v_i/u	0.703	0.66	0.653	0.652	0.652	0.652	0.652	0.652	0.652

This will lead to the power estimates compared with non-interacting turbines

$$\frac{P}{P_0} = \frac{1^3 + 0.7^3 + 8(0.65)^3}{10} = 0.35$$

So the efficiency id down to 35%. If the spacing is increased to x_0= 100m, we have the following wind speed ratios

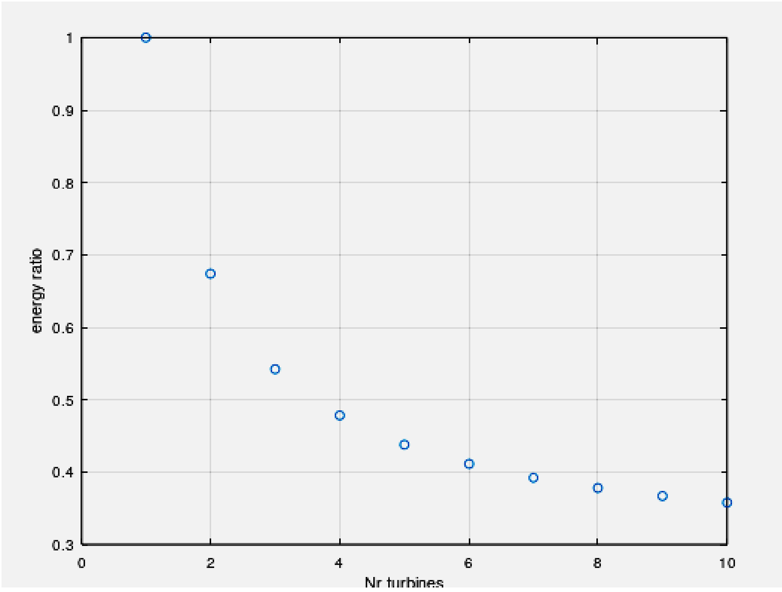
i	1	2	3	4	5	6	7	8	9
v_i/u	0.833	0.819	0.818	0.818	0.818	0.818	0.818	0.818	0.818

This will lead to the power estimates compared with non-interacting turbines,

$$\frac{P}{P_0} = \frac{1^3 + 0.83^3 + 8(0.82)^3}{10} = 0.60$$

which is almost double the efficiency. Clearly the turbine spacing x_0 relative to the turbine radius r_0 is crucial. Also very interesting is that the ratios become pretty much constant after the first couple of interacting turbines.

Here's a plot of total power ratios as a function of the number of turbines in a column, the ratio drops but appears to move towards a limit. The ratio for *N=100* is 0.285, and for *N=1000* is 0.278, and for *N=10,000* is 0.277 and for *N=100,000* is 0.277. So there's the limit.


energy ratio
Nr turbines
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
2
4
6
8
10